

On the Outage Capacity of Correlated Multiple-Path MIMO Channels

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Abstract

The use of multi-antenna arrays in both transmission and reception has been shown to dramatically increase the throughput of wireless communication systems. As a result there has been considerable interest in characterizing the ergodic average of the mutual information for realistic correlated channels. Here, an approach is presented that provides analytic expressions not only for the average, but also the higher cumulant moments of the distribution of the mutual information for zero-mean Gaussian MIMO channels with the most general multipath covariance matrices when the channel is known at the receiver. These channels include multi-tap delay paths, as well as general channels with covariance matrices that cannot be written as a Kronecker product, such as dual-polarized antenna arrays with general correlations at both transmitter and receiver ends. The mathematical methods are formally valid for large antenna numbers, in which limit it is shown that all higher cumulant moments of the distribution, other than the first two scale to zero. Thus, it is confirmed that the distribution of the mutual information tends to a Gaussian, which enables one to calculate the outage capacity. These results are quite accurate even in the case of a few antennas, which makes this approach applicable to realistic situations.

Index Terms

Wideband; Multipath; Beamforming; Capacity; Multiple Antennas; Random Matrix Theory; Replicas; Side Information

I. INTRODUCTION

Following pioneering work by [1], [2] it has become clear that the use of multi-antenna arrays in transmission and reception can lead to significantly increased bit-rates. This has led to a flurry of work calculating the narrowband ergodic mutual information of such systems, i.e. the mutual information averaged over realizations of the channel, using a variety of channel models and analytic techniques. For example, the ergodic capacity was calculated asymptotically for a large number of antennas, [3]–[12] or for large and small [10], [13] signal-to-noise ratios, using a variety of assumptions for the statistics [10], [14] of the fading channel.

To better understand the characteristics of realistic information transmission through fading channels, it is important to analyze the full distribution of the mutual information over realizations of fading. For example, the outage capacity [15] is sometimes a more realistic measure of capacity for delay constrained fading channels. In addition, the distribution of the mutual information provides information about the available diversity in the system [16]: the smaller the variance, the lower the probability of outage error when transmitting at a fixed rate. Finally, having an analytic expression for the distribution of the

mutual information allows one to simulate a system of multiple users in a simple way. [17] Recently, [8], [14] analytically calculated the first few moments of the distribution of the narrowband mutual information, asymptotically for large antenna numbers with spatial correlations. This analysis showed that the distribution is approximately Gaussian even for a few antennas, also seen in [17], [18]. More recently, other methods were devised to calculate all moments of the mutual information distribution exactly for some channel types. [19]–[24] Also, [10] calculated the ergodic mutual information in the large antenna limit for independent non-identically distributed (IND) channels, and extended their results to correlated channels with special restrictions on the correlations of different paths.

The above literature did not analyze the statistics of the mutual information for Gaussian channels with general non-Kronecker-product correlations. [25]–[28] These types of channels are becoming increasingly important to study, as it has recently been proposed that they appear in several situations, such as channels for generally correlated antennas with multiple polarizations. [27], [28]

Furthermore, the above works have generally focused on the case of narrowband flat-fading channels. However, the use of wide-band signals with non-trivial resolvable multipath necessitates the analysis of the mutual information in the presence of multipath. [29], [30] showed that the capacity of the wideband channel depends only on narrowband quantities, such as total average power etc. Subsequently, other authors have analyzed the wideband ergodic capacity using asymptotic methods. [7], [31] In a first attempt to describe the wideband distribution of the mutual information, [32] suggested that the distribution is Gaussian, if the number of independent paths is large. However, in many instances of interest the number of paths seen is small. [27], [33] It would thus be useful to analyze the effects of multi-path on the wideband mutual information of Gaussian MIMO fading channels of *arbitrary* multipath behavior in an analytic fashion. Although the exact methods mentioned above [19]–[24] can calculate all moments of the distribution for narrowband channels, they cannot be generalized to multi-path channels. Therefore, to make progress, one needs to rely on asymptotic methods.

In this paper we extend work done in [14] to provide analytic expressions for the statistics of the mutual information in the presence of multi-path with general spatially correlated channels. We assume that the instantaneous fading channel is known to the receiver but not the transmitter. Our results generalize the mutual information results of [10] for Gaussian channels to arbitrary zero-mean Gaussian correlated channels. The paths may or may not have the same delay. The methods used here apply the concept of replicas, which was initially introduced in statistical physics for understanding random systems [34], but in recent years have seen several applications in information theory. [4], [7], [35]–[37].

In particular, we obtain the following results:

- We use the replica method to calculate the moment generating function of the mutual information, averaging over general multipath, non-Kronecker product channels. Using this approach we derive expressions for its first three moments (mean, variance and skewness). As in [14] we find that for large antenna numbers n , the average of the distribution is of order n , the second moment of the distribution is of order unity, and the third moment is order $1/n$ respectively, while all other moments scale with higher powers of $1/n$. Thus, for large n the mutual information distribution approaches a Gaussian. Therefore, the outage mutual information can be expressed simply in terms of the mean and the variance of the distribution (section IV).
- We optimize the mean mutual information with respect to the input signal distribution to obtain the ergodic capacity (section IV-B).
- We demonstrate the dependence of the whole distribution of the mutual information on the specifics of the channel by calculating the mean and variance of the mutual information for a number of simple multipath channels.
- We also compare these Gaussian distributions with numerically generated ones and find very good agreement, even for a few antennas. This validates the analytical approach presented here for use in realistic situations with small antenna numbers.

A. Outline

In the remainder of this section we define relevant notation. In section II we describe the MIMO channels for which our method is applicable, in both the temporal and the frequency domain. In section III we define the wideband mutual information and in section III-A the statistics of its distribution. Subsequently, in section IV the mathematical framework of the method to calculate the generating function of the mutual information is presented. Also, the calculation of the ergodic capacity (section IV-B), its variance (section IV-C) and the higher order moments of the distribution (section IV-D) are discussed. Section IV-E deals with an alternative derivation of the results for the case when the receive correlation matrix is the same for all paths, while section IV-F briefly discusses the case of narrowband multipath, where all paths arrive at the same delay tap. In section V a few specific cases are analyzed analytically and compared to numerical Monte-Carlo calculations. Appendix I summarizes a number of complex integral identities employed in the main section, while Appendices II and III contain some details for various steps in section IV. Appendix IV includes some guiding details of the calculation of the higher order terms in section IV-D. Finally, Appendix V describes the procedure of evaluating the capacity-achieving transmission covariance \mathbf{Q} .

B. Notation

1) *Vectors/Matrices*: Throughout this paper, we will use bold-faced upper-case letters to denote matrices, e.g. \mathbf{X} , with elements given by X_{ab} , bold-faced lower-case letters for column vectors, e.g. \mathbf{x} with elements x_a , and non-bold letters for scalar quantities. Also the superscripts T and \dagger will indicate transpose and Hermitian conjugate operations and \mathbf{I}_n will represent the n -dimensional identity matrix.

Finally, the superscripts/subscripts t and r will be used for quantities referring to the transmitter and receiver, respectively.

2) *Gaussian Distributions*: The real Gaussian distribution with zero-mean and unit-variance will be denoted by $\mathcal{N}(0, 1)$, while the corresponding complex, circularly symmetric Gaussian distribution will be $\mathcal{CN}(0, 1)$.

3) *Order of Number of Antennas $\mathcal{O}(n^k)$* : We will be examining quantities in the limit when the number of transmitters n_t and number of receivers n_r , are both large but their ratios are fixed and finite. We will denote collectively the order in an expansion over the antenna numbers as $\mathcal{O}(n)$, $\mathcal{O}(1)$, $\mathcal{O}(1/n)$ etc., irrespective of whether the particular term involves n_t or n_r .

4) *Integral Measures*: Two general types of integrals over matrix elements will be dealt with and the following notation for their corresponding integration measures will be adopted. In the first type we will be integrating over the real and imaginary part of the elements of a complex $m_{rows} \times m_{cols}$ matrix \mathbf{X} . The integral measure will be denoted by

$$D\mathbf{X} = \prod_{a=1}^{m_{rows}} \prod_{\alpha=1}^{m_{cols}} \frac{d\text{Re}(X_{a\alpha}) d\text{Im}(X_{a\alpha})}{2\pi} \quad (1)$$

The second type of integration is over pairs of complex square matrices \mathcal{T} and \mathcal{R} . Each element of \mathcal{T} and \mathcal{R} will be integrated over a contour in the complex plane (to be specified). The corresponding measure will be described as

$$d\mu(\mathcal{T}, \mathcal{R}) = \prod_{a=1}^{m_{rows}} \prod_{\alpha=1}^{m_{cols}} \frac{d\mathcal{T}_{a\alpha} d\mathcal{R}_{a\alpha}}{2\pi i} \quad (2)$$

In addition, we will define a measure over a set of L pairs of matrices $\{\mathcal{T}^l, \mathcal{R}^l\}$ for $l = 0, \dots, L-1$ to be given simply by

$$d\mu(\{\mathcal{T}^l, \mathcal{R}^l\}) = \prod_{l=0}^{L-1} d\mu(\mathcal{T}^l, \mathcal{R}^l) \quad (3)$$

5) *Expectations*: We will use the notation $\langle \cdot \rangle$ to indicate an expectation over instantiations of the fading channel. We will reserve the notation $E[\cdot]$ for expectations over transmitted signals.

II. MULTIPATH MIMO CHANNEL MODEL

We consider the case of single-user transmission from n_t transmit antennas at a base station to n_r receive antennas at a mobile terminal over a fading channel with multiple paths with a finite bandwidth. We assume that the channel coefficients are known to the receiver, but not to the transmitter. The transmitted signal can be written in terms of discrete a time series representing the signals at discrete time steps $m\tau$ for $m \in \mathbf{Z}$ and τ the inverse available bandwidth. Thus we can use the following simple tap-delay model [29], [38]

$$\mathbf{y}_m = \sum_{l=0}^{L-1} \mathbf{G}_l \mathbf{x}_{m-m_l} + \mathbf{z}_m \quad (4)$$

where \mathbf{x}_m is the n_t -dimensional signal vector transmitted at time $m\tau$. Similarly, \mathbf{y}_m and \mathbf{z}_m are the corresponding n_r -dimensional received signal and noise vectors. \mathbf{z}_m is assumed an i.i.d. vector with each of its elements drawn from $\mathcal{CN}(0, 1)$, while \mathbf{G}_l is the $n_r \times n_t$ -dimensional complex channel matrix at delay times $m_l\tau$, where m_l is integer-valued. Of course, \mathbf{G}_l can be interpreted in a wider sense as an appropriately filtered version of the channel over the delay interval $(m_{l-1}\tau, m_l\tau]$. [38] Note that in general all paths need not arrive with different delays, i.e. we have $m_{l+1} \geq m_l$, with equality when the l th and $(l+1)$ th paths arrive within the same delay interval. In fact, all paths may be assumed to arrive over the same delay interval.

The analysis of multipath channels is simplified considerably by Fourier-transforming the transmitted and received signal vectors. In this case the Fourier-transformed received signal is solely a function of the corresponding Fourier component of the transmitted signal

$$\hat{\mathbf{y}}(\omega) = \hat{\mathbf{G}}(\omega) \hat{\mathbf{x}}(\omega) + \hat{\mathbf{z}}(\omega) \quad (5)$$

where the Fourier transform of the transmitter signal vector $\hat{\mathbf{x}}(\omega)$ is defined by

$$\hat{\mathbf{x}}(\omega) = \sum_{m=-\infty}^{\infty} e^{-i\omega m\tau} \mathbf{x}_m \quad (6)$$

with similar definitions for the Fourier components $\hat{\mathbf{y}}(\omega)$, $\hat{\mathbf{z}}(\omega)$. $\hat{\mathbf{G}}(\omega)$ is the Fourier transform of the channel impulse response given by

$$\hat{\mathbf{G}}(\omega) = \sum_{m=0}^{L-1} e^{-i\omega m\tau} \mathbf{G}_m \quad (7)$$

Note that (6) implies that each symbol vector $\hat{\mathbf{x}}(\omega)$ transmitted over a single frequency is spread over infinite times. As a result, it sees no interference from other frequency components due to multi-path. In practice, and in order to avoid mixing between close frequencies due to Doppler fading, one has to

transmit each symbol over a finite time window, therefore essentially using a discrete set of frequency components, e.g. $\omega_k = 2\pi k/(M\tau)$, with $k = 0, \dots, M-1$. The number of discrete frequency components M is usually chosen so that the symbol duration is less than the coherence time of the channel t_{coh} , i.e. $M < t_{coh}/\tau$. One can then send different symbols one after the other. However, there is a residual ISI interference due to multipath and the finite Fourier modes are no longer orthogonal. Various methods have devised to restore orthogonality, such as the inclusion of a cyclic prefix. [39] These issues will be ignored here and we will use the discrete Fourier mode version of (5) given by

$$\hat{\mathbf{y}}_{pk} = \hat{\mathbf{G}}_k \hat{\mathbf{x}}_{pk} + \hat{\mathbf{z}}_{pk} \quad (8)$$

where the index p represents the symbol index, k is the Fourier mode index with $k = 0, \dots, M-1$, $\hat{\mathbf{G}}_k$ is the corresponding channel Fourier component for $\omega_k = 2\pi k/(M\tau)$. $\hat{\mathbf{x}}_{pk}$ (and similarly $\hat{\mathbf{y}}_{pk}$, $\hat{\mathbf{z}}_{pk}$) have been normalized so that $\hat{\mathbf{z}}_k$, the Fourier transform of the noise vector \mathbf{z}_n is i.i.d. with elements $\sim \mathcal{CN}(0, 1)$. Also, the input signal in each frequency component $\hat{\mathbf{x}}_k$ is assumed Gaussian with covariance $E[\hat{\mathbf{x}}_k \hat{\mathbf{x}}_{k'}^\dagger] = \delta_{kk'} \mathbf{Q}_k$, normalized so that $\text{Tr}\{\mathbf{Q}_k\} = n_t$. For completeness, we rewrite the Fourier transform of the channel in (7) as

$$\hat{\mathbf{G}}_k = \sum_{l=0}^{L-1} \mathbf{G}_l e^{i \frac{2\pi k m_l}{M}} \quad (9)$$

As mentioned earlier, the channel matrices $\hat{\mathbf{G}}_k$ are assumed to be known at the receiver but not the transmitter.

A. Channel Statistics

Next, we would like to characterize the statistics of the channel matrices \mathbf{G}_l in (4), which are random due to fading. In particular, they are assumed to be zero-mean, independent Gaussian random matrices. In addition, we assume the correlations between elements of \mathbf{G}_l to be as follows:

$$\langle G_{l,i\alpha} G_{l',j\beta}^* \rangle = \delta_{ll'} \frac{\rho_l}{n_t} T_{l,ij} R_{l,\alpha\beta} \quad (10)$$

where the expectation $\langle \cdot \rangle$ is over the fading matrices \mathbf{G}_l . ρ_l , \mathbf{T}_l and \mathbf{R}_l are the signal to noise ratio, and the n_t - and n_r -dimensional correlation matrices for the l -th path at the transmitter and receiver, respectively. Underlying the structure of the above correlations is the assumption that different paths have uncorrelated channels. [40] Each path is assumed to have correlations in the form of a Kronecker product. This is certainly valid when each path corresponds to a single scattered wave, in which case each of the corresponding correlation matrices have unit rank. The above channel model is in agreement with adopted channel models in third generation standards [27].

We comment that an interesting special case occurs when *all* the delays m_l take the same value, i.e. arrive within the same τ interval (see section IV-F). This represents a *narrowband* channel with non-Kronecker product (or non-factorizable) correlations. In other words we could write the analogous simple narrowband channel relation

$$\mathbf{y}_p = \mathbf{G}\mathbf{x}_p + \mathbf{z}_p \quad (11)$$

where

$$\mathbf{G} = \sum_l \mathbf{G}_l \quad (12)$$

The matrices \mathbf{G}_l have correlations of the form (10) above. We note that such a form includes general models of polarization mixing with general correlation matrices between the different polarization components [27], [28]. For example, the correlations of a multipath channel with antennas of different polarizations can be written compactly as

$$\langle G_{i\alpha} G_{j\beta}^* \rangle = \sum_l \frac{\rho_l}{n_t} \begin{bmatrix} T_{\alpha\beta}^{l,v} \\ T_{\alpha\beta}^{l,h} \end{bmatrix}^T \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \begin{bmatrix} R_{ij}^{l,v} \\ R_{ij}^{l,h} \end{bmatrix} \quad (13)$$

where the sum is over all paths, ρ_l is the signal-to-noise ratio of each path l , $\mathbf{T}^{l,v}$, $\mathbf{T}^{l,h}$ are the correlation matrices of the vertical and horizontal polarization components of transmitter antennas for the l th path (and similarly for the receiver arrays) and x is the polarization mixing ratio.

Finally, it should be stressed that the most general narrowband zero-mean Gaussian model, including the recently proposed independent non-identically distributed (IND) channel, can be expressed in the form of (10), (12), since the correlations of any Gaussian zero-mean matrix can be written as

$$\langle G_{i\alpha} G_{j\beta}^* \rangle = \sum_l T_{l,ij} R_{l,\alpha\beta} \quad (14)$$

To see this, let $l = 1, \dots, (n_t n_r)^2$ and then set the matrices \mathbf{T}_l , \mathbf{R}_l have zero entries except for the element ij and $\alpha\beta$, respectively, when the index l takes the value $l(i, j, \alpha, \beta) = i + n_t(j-1) + n_t^2(\alpha-1) + n_t^2 n_r(\beta-1)$. The non-zero values of these matrices can be chosen to be, for example, $T_{l(i,j,\alpha,\beta),ij} = \langle G_{i\alpha} G_{j\beta}^* \rangle$ and $R_{l(i,j,\alpha,\beta),\alpha\beta} = 1$. Although this mapping is not unique, it demonstrates the generality of our method.

It should be noted that, since the receiver/mobile terminal is usually assumed to be located deep inside the clutter, the received signal tends to have wide angle-spread, thereby making the differences in the angles of arrival of different paths less distinguishable. Therefore, it is sometimes reasonable to assume that the receiver correlations \mathbf{R}_l are path-independent, i.e.

$$\langle G_{l,i\alpha} G_{l',j\beta}^* \rangle = \delta_{ll'} \frac{\rho_l}{n_t} T_{l,ij} R_{\alpha\beta} \quad (15)$$

This assumption is not as easily met at the transmitter/base station, where the nearest scatterers are typically further separated, thereby making the \mathbf{T}_l typically different. A further simplification of the above is the case when the receive antennas are uncorrelated, which is discussed in [29].

As a result of the above, $\hat{\mathbf{G}}_k$, the Fourier transform of \mathbf{G}_l (9) is also Gaussian with correlations

$$\left\langle \hat{\mathbf{G}}_{k,i\alpha} \hat{\mathbf{G}}_{k',j\beta}^* \right\rangle = \frac{1}{n_t} \sum_{l=0}^{L-1} \rho_l e^{i \frac{2\pi(k-k')m_l}{M}} T_{l,ij} R_{l,\alpha\beta} \quad (16)$$

For the case of narrowband channels mentioned above in (11), (12), $\hat{\mathbf{G}}_k$ is nonzero only for $k = 0$, therefore $\hat{\mathbf{G}}_0 = \mathbf{G}$ with \mathbf{G} given in (12).

III. WIDEBAND MUTUAL INFORMATION

The mutual information of each of the frequency components k is given by [1], [2]

$$I_k = \log \det \left(\mathbf{I}_{n_r} + \hat{\mathbf{G}}_k \mathbf{Q}_k \hat{\mathbf{G}}_k^\dagger \right) \quad (17)$$

The log above (and throughout the whole paper) represents the natural logarithm and thus I is expressed in nats. The total mutual information over all frequency components is then

$$I = \sum_{k=0}^{M-1} I_k \quad (18)$$

A. Statistics of Mutual Information

The distribution of the mutual information can be characterized through its moments. These moments can be evaluated by first calculating the moment generating function $g(\nu)$ of I

$$g(\nu) = \left\langle \left[\prod_{k=0}^{M-1} \det \left(\mathbf{I}_{n_r} + \hat{\mathbf{G}}_k \mathbf{Q}_k \hat{\mathbf{G}}_k^\dagger \right) \right]^{-\nu} \right\rangle \quad (19)$$

$$\begin{aligned} &= \langle e^{-\nu I} \rangle \\ &= 1 - \nu \langle I \rangle + \frac{\nu^2}{2} \langle I^2 \rangle + \dots \end{aligned} \quad (20)$$

Assuming that $g(\nu)$ is analytic at least in the vicinity of $\nu = 0$, we can express $\log g(\nu)$ as follows

$$\log g(\nu) = \sum_{p=1}^{\infty} \frac{(-\nu)^p}{p!} \mathcal{C}_p \quad (21)$$

where \mathcal{C}_p is the p -th cumulant moment of I . For example, the ergodic mutual information, i.e. the average of the distribution is given by

$$\mathcal{C}_1 = \langle I \rangle = \sum_{k=0}^{M-1} \langle I_k \rangle \quad (22)$$

$$= \sum_{k=0}^{M-1} \left\langle \log \det \left(\mathbf{I}_{n_r} + \hat{\mathbf{G}}_k \mathbf{Q}_k \hat{\mathbf{G}}_k^\dagger \right) \right\rangle \quad (23)$$

Similarly, the variance of the distribution is

$$\mathcal{C}_2 = \text{Var}(I) = \langle (I - \langle I \rangle)^2 \rangle \quad (24)$$

$$= \left(\sum_{k,k'=0}^{M-1} \langle I_k I_{k'} \rangle \right) - \langle I \rangle^2 \quad (25)$$

its the skewness of the distribution is

$$\mathcal{C}_3 = \text{Sk}(I) = \langle (I - \langle I \rangle)^3 \rangle \quad (26)$$

and so forth. Note that since I_k depends only on $\hat{\mathbf{G}}_k$ and \mathbf{Q}_k , to evaluate the ergodic average (23) we can perform the average for each term in the sum in (23) separately, neglecting any correlations between $\hat{\mathbf{G}}_k$'s with different k indices. Thus for evaluating the ergodic average, the only correlation of relevance is $\langle \hat{G}_{k,i\alpha} \hat{G}_{k,j\beta} \rangle$ which turns out to be k -independent, as seen in (16). Therefore the only k -dependence of each term in the sum in (23) is through \mathbf{Q}_k . As a result the optimal \mathbf{Q}_k will be k -independent. We will thus henceforth assume that \mathbf{Q}_k is chosen to be a k -independent quantity \mathbf{Q} . As a result, the wideband ergodic capacity becomes just M times its narrowband counterpart [29]. This k -independence of the mean mutual information will be of use in the next section. In contrast, in evaluating higher moments of the distribution such as the variance, as is easily seen from (25), we will have to consider cross correlations between $\hat{\mathbf{G}}_k$ and $\hat{\mathbf{G}}_{k'}$

Finally, it should be emphasized that the distribution of the mutual information can also be completely characterized by the outage mutual information [15], obtained by inverting the expression below with respect to I_{out}

$$P_{out} = \text{Prob}(I < I_{out}) \quad (27)$$

where $\text{Prob}(I < I_{out})$ is the probability that the mutual information is less than a given value I_{out} .

IV. MATHEMATICAL FRAMEWORK

The purpose of this paper is to analyze the statistics of the wideband mutual information I in (18) for general zero-mean Gaussian channels. In this section we describe the basic steps to derive analytic expressions for the first few cumulant moments of I , valid formally for large antenna numbers. In this limit it has been shown elsewhere [14], [17], [18], [41] that the narrowband mutual information distribution becomes asymptotically Gaussian. Thus the first two moments can describe the outage mutual information (27). Using the mathematical framework of [8], [14] we will show that this Gaussian character holds also for wideband channels.

To obtain the moments of the mutual information distribution we need to calculate $g(\nu)$ in (19) for ν in the vicinity of $\nu = 0$. To achieve this we will employ the replica assumption discussed in [4], [8], [14], [42].

Assumption 1 (Replica Method): $g(\nu)$ evaluated for positive integer values of ν can be analytically continued for real ν , specifically in the vicinity of $\nu = 0^+$.

This assumption, used also in [7], [36], [37], [43], [44], alleviates the problem of dealing with averages of logarithms of random quantities, since the logarithm is obtained after calculating $g(\nu)$.

In Appendix II we show that $g(\nu)$ can be expressed as an integral over $M\nu \times M\nu$ complex matrices $\mathcal{R}^l, \mathcal{T}^l$, with $l = 0, \dots, L - 1$

$$g(\nu) = \int d\mu \left(\{\mathcal{T}^l, \mathcal{R}^l\} \right) e^{-\mathcal{S}} \quad (28)$$

where the integration metric was defined in (2) and

$$\begin{aligned} \mathcal{S} &= \log \det \left(\mathbf{I}_{n_r} \otimes \mathbf{I}_{\nu M} + \sum_l \frac{1}{\sqrt{n_t}} \mathbf{R}_l \otimes \mathcal{R}^l \right) \\ &+ \log \det \left(\mathbf{I}_{n_r} \otimes \mathbf{I}_{\nu M} + \sum_l \frac{\rho_l}{\sqrt{n_t}} \mathbf{Q} \mathbf{T}_l \otimes \hat{\mathcal{T}}^l \right) \\ &- \sum_l \text{Tr} \left\{ \mathcal{T}^l \mathcal{R}^l \right\} \end{aligned} \quad (29)$$

where $\hat{\mathcal{T}}^l$ is an $M\nu \times M\nu$ matrix related to \mathcal{T}^l via

$$\hat{\mathcal{T}}_{k\alpha; k'\beta}^l = \mathcal{T}_{k\alpha; k'\beta}^l e^{\frac{2\pi i(k-k')m_l}{M}} \quad (30)$$

where we have explicitly written out the components of the matrices here with k, k' ranging from 0 to $M - 1$ and α and β ranging from 1 to ν . (See the notation in Appendix II).

At this point ν is still a positive integer, which has to be taken to zero following Assumption 1, in order to be able to expand $g(\nu)$ for small ν , as in (20). However, since the integral in (28) cannot be performed exactly, we need to calculate it asymptotically in the limit of large antenna numbers $n_t, n_r \gg 1$. Therefore we need to interchange the limits $n \gg 1$ and $\nu \rightarrow 0^+$.

Assumption 2 (Interchanging Limits): [14] The limits $n \rightarrow \infty$ and $\nu \rightarrow 0^+$ in evaluating $g(\nu)$ in (28) can be interchanged by first taking the former and then the latter without affecting the final answer.

A. Saddle-Point Analysis

We now use Assumption 2 to calculate (28) asymptotically for large n_t, n_r , by deforming the integrals in (28) to pass through a saddle point. More details are given in [14]. To specify the structure of the

saddle-point solution, i.e. the form of $\mathcal{T}^l, \mathcal{R}^l$ at the saddle-point, we assume as in [14] that the relevant saddle-point solution is invariant in ν -dimensional replica space. However, in our case since $\mathcal{T}^l, \mathcal{R}^l$ are νM -dimensional matrices, this is not enough to fully characterize the saddle-point. Therefore, we will also assume that the saddle-point values of $\mathcal{T}^l, \mathcal{R}^l$ are invariant in M -dimensional frequency q -space. This assumption, as we shall see, leads to a saddle-point value of \mathcal{S} , and to an ergodic average of the mutual information, that is independent of inter-frequency correlations, in agreement with the correct answer, as discussed in Section III-A and [29].

Thus, at the saddle-point $\mathcal{T}^l, \mathcal{R}^l$ take the form $\mathcal{T}^l = t_l \sqrt{n_t} \mathbf{I}_{\nu M}, \mathcal{R}^l = r_l \sqrt{n_t} \mathbf{I}_{\nu M}$, where t_l and r_l are positive, still undetermined numbers of order unity in the number of antennas. A scaling factor of $\sqrt{n_t}$ has been included for convenience, as will become evident below. Following [14] we analyze the integral in (28) by shifting the origin of integration to the saddle point, i.e. by rewriting \mathcal{T}, \mathcal{R} as

$$\mathcal{T}^l = t_l \sqrt{n_t} \mathbf{I}_{\nu M} + \delta \mathcal{T}^l \quad (31)$$

$$\mathcal{R}^l = r_l \sqrt{n_t} \mathbf{I}_{\nu M} + \delta \mathcal{R}^l$$

where $\delta \mathcal{T}^l, \delta \mathcal{R}^l$ are νM -dimensional matrices representing deviations around the saddle point. One can then expand \mathcal{S} in (29) in a Taylor series of increasing powers of $\delta \mathcal{T}^l, \delta \mathcal{R}^l$ as follows

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 \dots \quad (32)$$

with \mathcal{S}_p containing p -th order terms in $\delta \mathcal{T}^l, \delta \mathcal{R}^l$. These terms are shown explicitly in Appendix III in (90), (91), (92), (95), where it can be seen that \mathcal{S}_p is $\mathcal{O}(n^{1-p/2})$, making (32) indeed an asymptotic expansion in inverse powers of n .

The saddle point solution of (28) and hence the corresponding values of t_l, r_l is found by demanding that \mathcal{S} is stationary with respect to variations in $\mathcal{T}^l, \mathcal{R}^l$. [45] This means that $\mathcal{S}_1 = 0$ (see (91)), which is analogous to setting the first derivative of a function to zero, in order to find its maximum or minimum. This produces the following saddle-point equations:

$$r_l = \frac{\rho_l}{n_t} \text{Tr} \left\{ \mathbf{Q} \mathbf{T}_l \left[\mathbf{I}_{n_t} + \mathbf{Q} \tilde{\mathbf{T}} \right]^{-1} \right\} \quad (33)$$

$$t_l = \frac{1}{n_t} \text{Tr} \left\{ \mathbf{R}_l \left[\mathbf{I}_{n_r} + \tilde{\mathbf{R}} \right]^{-1} \right\} \quad (34)$$

where $\tilde{\mathbf{T}}, \tilde{\mathbf{R}}$ have been defined as

$$\tilde{\mathbf{T}} = \sum_l \rho_l t_l \mathbf{T}_l \quad (35)$$

$$\tilde{\mathbf{R}} = \sum_l r_l \mathbf{R}_l \quad (36)$$

The next term in the expansion of \mathcal{S} is \mathcal{S}_2 and needs to be taken into account non-perturbatively, because it is $\mathcal{O}(1)$ in the number of antennas n and thus will provide a finite correction. Fortunately, \mathcal{S}_2 is quadratic in the variables $\delta\mathcal{T}^l$ and $\delta\mathcal{R}^l$ so that the integral (28) is just a Gaussian integral at this order.

In contrast, \mathcal{S}_p terms with $p > 2$ become vanishingly small at large n , since they are $\mathcal{O}(n^{1-p/2})$. Therefore, they can be expanded from the exponent in (28) and treated perturbatively as follows:

$$g(\nu) = e^{-\mathcal{S}_0} \int d\mu \left(\{\delta\mathcal{T}^l\}, \{\delta\mathcal{R}^l\} \right) e^{-\mathcal{S}_2} \cdot \left(1 - \mathcal{S}_3 - \mathcal{S}_4 + \frac{1}{2}\mathcal{S}_3^2 + \dots \right) \quad (37)$$

Each term in this expansion can be evaluated explicitly, with higher order terms producing corrections of increasingly higher orders in $1/n$. Subsequently, taking the logarithm of the result as prescribed in (21) will produce an $1/n$ -expansion for the cumulant moments of I , with only integer powers of $1/n$ surviving [14]).

B. Ergodic Capacity

From (90) in Appendix III we see that $\mathcal{S}_0 = \nu\Gamma$ with proportionality factor Γ being the leading term to the mutual information, which is given by

$$\begin{aligned} \Gamma &= M \log \det \left(\mathbf{I}_{n_t} + \mathbf{Q}\tilde{\mathbf{T}} \right) \\ &+ M \log \det \left(\mathbf{I}_{n_r} + \tilde{\mathbf{R}} \right) - n_t M \sum_l r_l t_l \end{aligned} \quad (38)$$

where $t_l, r_l, \tilde{\mathbf{T}}, \tilde{\mathbf{R}}$ are given by (34), (33), (35), (36).

Note that the above equations are independent of the relative delays between paths, thereby applying to narrowband channels, as well as wideband channels with non-trivial delays between paths. This is to be expected since the ergodic wideband capacity is independent of delay. [29]

To obtain the capacity-achieving input distribution \mathbf{Q} , $\langle I \rangle$ has to be optimized subject to the power constraint $\text{Tr}\{\mathbf{Q}\} = n_t$. This constraint is enforced by adding a Lagrange multiplier to $\langle I \rangle$, i.e.

$$\begin{aligned} \langle I \rangle &\rightarrow \langle I \rangle - \Lambda (\text{Tr}\{\mathbf{Q}\} - n_t) \\ &= \langle I \rangle - \Lambda \left(\sum_i q_i - n_t \right) \end{aligned} \quad (39)$$

where q_i are the n_t eigenvalues of \mathbf{Q} . As in [14], the eigenvectors of the optimal \mathbf{Q} are the same as $\tilde{\mathbf{T}}$ (at least to $\mathcal{O}(1/n)$). This statement is proven in Appendix V.

With the constraint that \mathbf{Q} and $\tilde{\mathbf{T}}$ should be diagonal in the same basis, we can find the optimal \mathbf{Q} by differentiating with respect to the eigenvalues q_i . It is then easy to see [14] that the optimal eigenvalues of \mathbf{Q} are given by

$$q_i = \left[\frac{1}{\Lambda} - \frac{1}{\tilde{T}_i} \right]_+ \quad (40)$$

where \tilde{T}_i are the n_t eigenvalues of $\tilde{\mathbf{T}}$ and $[x]_+ = \{x + \text{sgn}(x)\}/2$. Here, the Lagrange multiplier $\Lambda > 0$ is determined by imposing the power constraint

$$\text{Tr} \{ \mathbf{Q} \} = \sum_{i=1}^{n_t} q_i = n_t \quad (41)$$

with q_i given by (40).

C. Variance of the Mutual Information

To obtain the $\mathcal{O}(\nu^2)$ term in the expansion of $\log g(\nu)$ in (21) we need to only include the next non-vanishing term, \mathcal{S}_2 . The second line in (37) can be temporarily neglected.

Using the saddle point value for $\mathcal{S}_0 = \nu\Gamma$ in Eq.(37), the integration over $\delta\mathcal{R}^l$, $\delta\mathcal{T}^l$ can be performed straightforwardly (see [14] for more details), resulting in

$$g(\nu) = e^{-\nu\Gamma} \prod_{k,k'} \left| \det \mathbf{V}^{kk'} \right|^{-\frac{\nu^2}{2}} \quad (42)$$

where the $2L$ -dimensional matrix $\mathbf{V}^{kk'}$ is given in Appendix III by (94). Thus, by comparing (21) to (42) and by matching order by order the terms of the ν -Taylor expansion of $\log g(\nu)$, the leading term in the variance of the mutual information is

$$\begin{aligned} \mathcal{C}_2 &= \text{Var}(I) = - \sum_{kk'} \log \left| \det \mathbf{V}^{kk'} \right| + \mathcal{O}(1/n^2) \\ &= - \sum_{kk'} \log \det \left(\mathbf{I}_L - \mathbf{M}_{r,2}^{1/2} \mathbf{M}_{t,2} \mathbf{M}_{r,2}^{1/2} \right) + \mathcal{O}(1/n^2) \end{aligned} \quad (43)$$

where the L -dimensional matrices $\mathbf{M}_{t,2}$, $\mathbf{M}_{r,2}$ are given in Appendix III by (96) and (97). We note that since $\mathbf{M}_{r,2}$ and $\mathbf{M}_{t,2}$ are both $\mathcal{O}(1)$, the variance is also formally $\mathcal{O}(1)$ in the $1/n$ expansion when both n_t and n_r are of the same order.

D. Higher Order Terms

To obtain higher-order corrections in n^{-1} , beyond the $\mathcal{O}(n)$ and $\mathcal{O}(1)$ terms that appear in the average and the variance, respectively, one needs to take into account the terms \mathcal{S}_p with $p > 2$ in (37). These terms will give rise to higher-order cumulant moments of the distribution of the mutual information,

as well as higher-order corrections to the first two cumulant moments. In Appendix IV we sketch the calculation of the next leading correction terms of order $\mathcal{O}(n^{-1})$. Including this additional term $g(\nu)$ can be written as

$$g(\nu) = e^{-\nu\Gamma} \prod_{k,k'} \left| \det \mathbf{V}^{kk'} \right|^{-\nu^2/2} [1 + D_1 + \mathcal{O}(n^{-2})] \quad (44)$$

where D_1 is given by

$$D_1 = a_1\nu + a_3\nu^3 \quad (45)$$

and a_1 and a_3 are defined in (106), (107), which are indeed $\mathcal{O}(1/n)$.

Using the cumulant expansion notation of (21) and matching the generated terms above to the appropriate powers of ν , we see that D_1 produces order $\mathcal{O}(1/n)$ terms to the first cumulant (mean) \mathcal{C}_1 and third cumulant (skewness) \mathcal{C}_3 :

$$\mathcal{C}_1 = \Gamma - a_1 + \mathcal{O}(1/n^3) \quad (46)$$

$$\mathcal{C}_3 = -6a_3 + \mathcal{O}(1/n^3) \quad (47)$$

E. Special Case 1: \mathbf{R}_l independent of l

In this section, we will show how the above results simplify when the correlation matrix of the receiver or transmitter is independent of the path index l . For concreteness we will only analyze the case where \mathbf{R}_l is independent of l , i.e. when the channel correlations take the form (15).

In this case we see that in (34) t_l is independent of the path index l and thus we may set $t = t_l$. Furthermore, by summing (33) over l we get

$$r \equiv \sum_l r_l = \frac{1}{n_t} \text{Tr} \left\{ \frac{\mathbf{Q}\mathbf{T}}{\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}} \right\} \quad (48)$$

$$t = \frac{1}{n_r} \text{Tr} \left\{ \frac{\mathbf{R}}{\mathbf{I}_{n_r} + r\mathbf{R}} \right\} \quad (49)$$

where $\mathbf{T} = \sum_l \rho_l \mathbf{T}_l$ and $\mathbf{R} = \mathbf{R}_l$. Thus the mutual information in (38) may be written as

$$\begin{aligned} \langle I \rangle &= M \log \det (\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}) \\ &+ M \log \det (\mathbf{I}_{n_r} + r\mathbf{R}) - Mn_t r t \end{aligned} \quad (50)$$

Note that, apart from a redefinition of \mathbf{T} to take into account multiple paths, these results are identical to those derived previously for narrowband channels [14].

To derive a simplified expression for the variance from (43), we note that $M_{r,2}$ now becomes a constant matrix, which can be written as a vector outer product

$$\mathbf{M}_{r,2}^{ll'} = \frac{1}{n_t} \text{Tr} \left\{ \left(\frac{\mathbf{R}}{\mathbf{I}_{n_r} + r\mathbf{R}} \right)^2 \right\} \mathbf{v}\mathbf{v}^\dagger = m_{r,2} \mathbf{v}\mathbf{v}^\dagger \quad (51)$$

where the vector \mathbf{v} has elements $v_l = 1$ for all $l = 1, \dots, L$. The second equality in the above equation defines $m_{r,2}$. Similarly, $M_{t,2}$ can be written as:

$$M_{t,2}^{ll'} = \frac{\rho_l \rho_{l'}}{n_t} \exp \left[\frac{2\pi i (k_1 - k_2) (m_l - m_{l'})}{M} \right] \cdot \text{Tr} \left\{ \frac{1}{\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}} \mathbf{Q}\mathbf{T}_l \frac{1}{\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}} \mathbf{Q}\mathbf{T}_{l'} \right\} \quad (52)$$

After some algebra we see that (43) simplifies to

$$\text{Var}(I) = - \sum_{kk'} \log \left| 1 - m_{r,2} m_{t,2}^{k-k'} \right| \quad (53)$$

where

$$m_{t,2}^q = \frac{1}{n_t} \text{Tr} \left\{ \frac{1}{\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}} \mathbf{Q}\mathbf{S}_q \frac{1}{\mathbf{I}_{n_t} + t\mathbf{Q}\mathbf{T}} \mathbf{Q}\mathbf{S}_{-q} \right\} \quad (54)$$

with the matrix \mathbf{S}_q defined as

$$\mathbf{S}_q = \sum_l \rho_l \mathbf{T}_l \exp \left(\frac{2\pi i q m_l}{M} \right) \quad (55)$$

which is the temporal Fourier transform of the correlation matrices \mathbf{T}_l .

F. Special Case 2: Narrowband Multipath

As mentioned in the introduction, this approach is applicable in calculating the ergodic average and variance of an arbitrary Gaussian zero-mean channel. This obviously includes a narrowband channel with arbitrary correlations. The only difference in the analysis of this channel is that all delay indices m_l are equal and can thus be set to zero.

V. ANALYSIS OF RESULTS

In the previous section we have seen that in the limit of large antenna numbers n , the mean mutual information is of order n , while the variance of the distribution is of order unity. In addition, in Appendix IV we find that the skewness (the third cumulant moment) is $\mathcal{O}(1/n)$ and higher cumulant moments are even smaller ($\mathcal{O}(1/n^2)$). In agreement with the narrowband case [14], [17], this suggests that the distribution of the wideband multipath mutual information is also Gaussian for large n . This Gaussian

behavior was seen to be very accurate even for small antenna arrays for narrowband channels [14], [18]. Below, we will see this to hold also in wideband multipath channels by numerically comparing the Gaussian distribution $\mathcal{N}[\langle I \rangle, \text{Var}(I)]$ calculated using (90) and (43) with the simulated distribution resulting from the generation of a large number of random matrix realizations. We will specifically analyze four representative situations to show the effects of multipath on the distribution of the mutual information of wideband channels.

If the distribution of the mutual information is Gaussian, we can express I_{out} from (27) as

$$I_{out} = \langle I \rangle - \sqrt{2\text{Var}(I)}\Phi^{-1}(2P_{out} - 1) \quad (56)$$

where $\Phi^{-1}(x)$ is the inverse error function. [46] Clearly, this can only be an approximation, since the mutual information cannot take negative values.

A. Distribution of Wideband Mutual Information for L equal-power equally-spaced i.i.d. paths

It is instructive to apply the above results to the case of L equal power paths, with $\rho_l = \rho/L$ in (16), with $n_t = n_r = n$ and with correlation matrices being unity, i.e. $\mathbf{R}_l = \mathbf{T}_l = \mathbf{I}_n$. Also, for simplicity we assume the delays of the paths are all equally spaced from each other by τ , i.e. $m_l = l$. This is a special case of the one discussed in Section IV-E. In this case the optimal input distribution is $\mathbf{Q} = \mathbf{I}_n$ [14], and (50) becomes

$$\langle I \rangle = nM [\log(1 + \rho t) + \log(1 + r) - tr] \quad (57)$$

with the extremizing values of r and t from (48), (49) given by

$$r = \rho t = \frac{\sqrt{1 + 4\rho} - 1}{2} \quad (58)$$

which gives

$$\langle I \rangle = nM \left[2 \log \left(\frac{\sqrt{1 + 4\rho} + 1}{2} \right) - \frac{(\sqrt{1 + 4\rho} - 1)^2}{4\rho} \right] \quad (59)$$

This result is identical to the one derived elsewhere [14], [47]. The variance can be calculated using (53), (54) with the \mathbf{S}_q in (55) taking the form $S_q = \rho \mathbf{I}_n / L$ and takes the form

$$\text{Var}(I) = - \sum_{k,k'=0}^{M-1} \log \left[1 - \left(\frac{t\rho}{t\rho + 1} \frac{\sin \frac{\pi L(k-k')q}{M}}{L \sin \frac{\pi(k-k')q}{M}} \right)^2 \right] \quad (60)$$

with t given by (58). We see that the larger L is, the more peaked the ratio inside the logarithm is, and therefore the smaller the variance. If $L = M$, the ratio of sines in (43) becomes proportional to a

Kronecker delta function $\delta_{kk'}$, so that the variance becomes equal to

$$\text{Var}(I) = -M \log \left[1 - \left(\frac{\sqrt{1+4\rho} - 1}{\sqrt{1+4\rho} + 1} \right)^2 \right] \quad (61)$$

In general we can say that the variance of the *normalized* mutual information per channel (i.e. I/M) scales as $\text{Var}(I/M) \sim 1/L$.

B. Distribution of Wideband Mutual Information for an exponentially distributed power delay profile

We can also apply this approach to a more realistic version of a multipath channel, namely one with an exponential power delay profile, which can be expressed as

$$\rho_l = \bar{\rho} \left(1 - e^{-\delta} \right) e^{-\delta l} \quad (62)$$

where $\delta^{-1} = d/\tau$ is the product of the delay constant d with the bandwidth τ^{-1} , and $\bar{\rho}$ is the signal-to-noise ratio for the total power-delay profile. We have implicitly assumed here that the number of paths is infinite, $L = \infty$. For the simple case of uncorrelated channels, where both \mathbf{T}^l and \mathbf{R}^l are unit matrices, the average mutual information is identical to (57), by replacing ρ with $\bar{\rho}$. This can easily be seen by observing that the average mutual information in (50) is a function of ρ_l only through \mathbf{T} , which here is equal to

$$\mathbf{T} = \sum_l \rho_l \mathbf{T}_l = \mathbf{I}_{n_t} \bar{\rho} \quad (63)$$

To calculate the variance of I , we first need to calculate $m_{t,2}^{k-k'}$ and $m_{r,2}$ in (53). The former can be evaluated from (54) by performing the sum (55)

$$\begin{aligned} \mathbf{S}_q &= \sum_{l=0}^{\infty} \rho_l \mathbf{T}_l e^{\frac{2\pi i q m_l}{M}} = \mathbf{I}_{n_t} \sum_{l=0}^{\infty} \rho_l e^{\frac{2\pi i q m_l}{M}} \\ &= \bar{\rho} \frac{1 - e^{-\delta}}{1 - e^{-\delta + \frac{2\pi q i}{M}}} \end{aligned} \quad (64)$$

As a result $m_{t,2}^q$ (and similarly $m_{r,2}$) can be expressed as

$$m_{t,2}^q = \frac{\bar{\rho}^2}{(1 + t\bar{\rho})^2} \left| \frac{1 - e^{-\delta}}{1 - e^{-\delta + 2\pi i q/M}} \right|^2 \quad (65)$$

$$m_{r,2} = \frac{1}{(1 + r)^2} = t^2 \quad (66)$$

so that the normalized variance per channel can be expressed as

$$\text{Var} \left(\frac{I}{M} \right) = -\frac{1}{M^2} \sum_{k,k'} \log \left| 1 - m_{r,2} m_{t,2}^{k-k'} \right| \quad (67)$$

When the number of frequency channels M is large, we can approximate the above sums with integrals over frequency, which can be performed analytically to give

$$\begin{aligned} \text{Var} \left(\frac{I}{M} \right) &= -\ln \left\{ \frac{1}{2} \left[1 + e^{-2\delta} - \beta (1 - e^{-\delta})^2 \right. \right. \\ &\quad \left. \left. + \sqrt{\left(1 + e^{-2\delta} - \beta (1 - e^{-\delta})^2 \right)^2 - 4e^{-2\delta}} \right] \right\} \end{aligned} \quad (68)$$

where

$$\beta = \left(\frac{t\bar{\rho}}{1 + t\bar{\rho}} \right)^2 = \frac{16\bar{\rho}^2}{(1 + \sqrt{1 + 4\bar{\rho}})^4} \quad (69)$$

(68) is plotted in Fig. 1 as a function of the delay.

C. Interdependence of spatial and temporal correlations

In the previous section, we analyzed the situation where all paths had the same transmission correlation matrices $\mathbf{T}_l = \mathbf{I}_{n_t}$ resulting to significant simplifications. This situation is not necessarily realistic. Typically, each path has an angle spread smaller than the composite angle-spread and with a different mean angle of departure from the transmitter for each path. [27] Thus, even if the composite narrowband correlations at the transmitter are assumed to be low, the associated correlations per path may be substantial. It is therefore interesting to compare the mutual information distribution of the following two situations: In the first, all paths have a correlation matrix identical to the narrowband composite correlation matrix. In the second, each path has different correlation matrices, subject to giving the same narrowband correlation matrix as in the first case. For simplicity we will take the narrowband composite correlation matrix to be unity, with the following correlations between transmitting antennas:

$$T_{ab} = \int_{-180}^{180} \frac{d\phi}{\sqrt{2\pi\delta^2}} e^{2\pi i(a-b)d_\lambda \sin((\phi+\phi_0)\pi/180) - \phi^2/(2\delta^2)} \quad (70)$$

with $a, b = 1 \dots n_t$ being the index of transmitting antennas. This is a simple model for the antenna correlations of a uniform linear ideal antenna array with $d_\lambda = d_{min}/\lambda$ the nearest neighbor antenna spacing in wavelengths, a Gaussian power azimuth spectrum with angle-spread δ degrees and ϕ_0 degrees mean angle of departure. [48], [49]

In Fig. 2 we see that, although the mean mutual information is identical in the cases, the variance of the mutual information of the second case is roughly double to that of the first case. We thus see that the correlation structure of the underlying paths have a significant effect on the mutual information distribution.

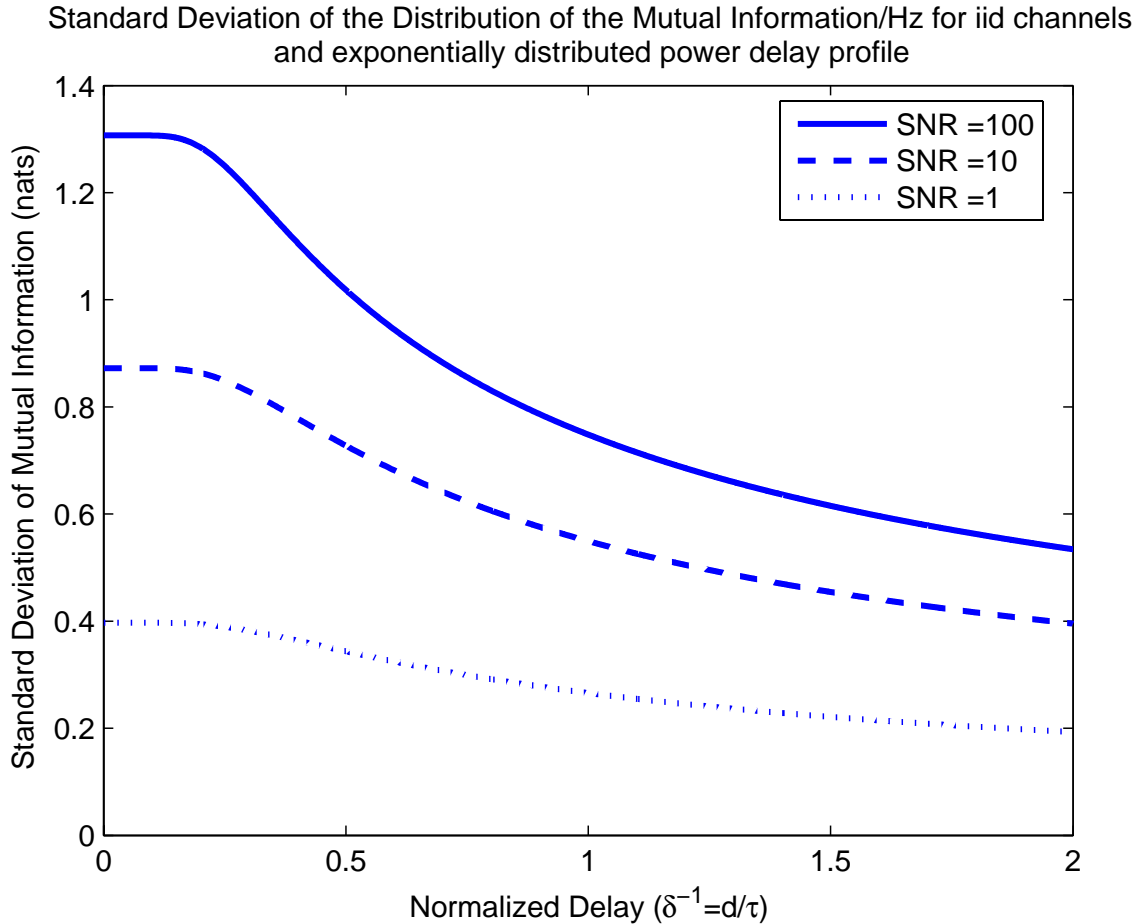


Fig. 1. Standard deviation of the mutual information as a function of the normalized delay spread (d/τ) for the case of an exponential power delay profile for three different signal-to-noise ratios. For zero delay ($d = 0$), the narrowband result is recovered (y -axis). For increasing delays compared to bandwidth $d > \tau$, the standard deviation of the mutual information decreases. Eq. (68) has been used.

D. Example: L distinct fully correlated paths

As a final example, we describe a simple version of the general non-Kronecker channel case given by (10). In particular, we assume that $n_t = n_r = n$ and that the correlation matrices $\mathbf{T}_l, \mathbf{R}_l$ are mutually orthogonal, rank-one matrices, e.g., for the transmitter we have $\mathbf{T}_l = n\mathbf{a}_l\mathbf{a}_l^\dagger$, with $\mathbf{a}_l^\dagger\mathbf{a}_{l'} = \delta_{ll'}$. This corresponds to a set of $L \leq n$ orthogonal plane-waves at the transmitter, each of which are connected with a plane-wave arriving at the receiver in orthogonal directions. In this case, (33) and (34) simplify

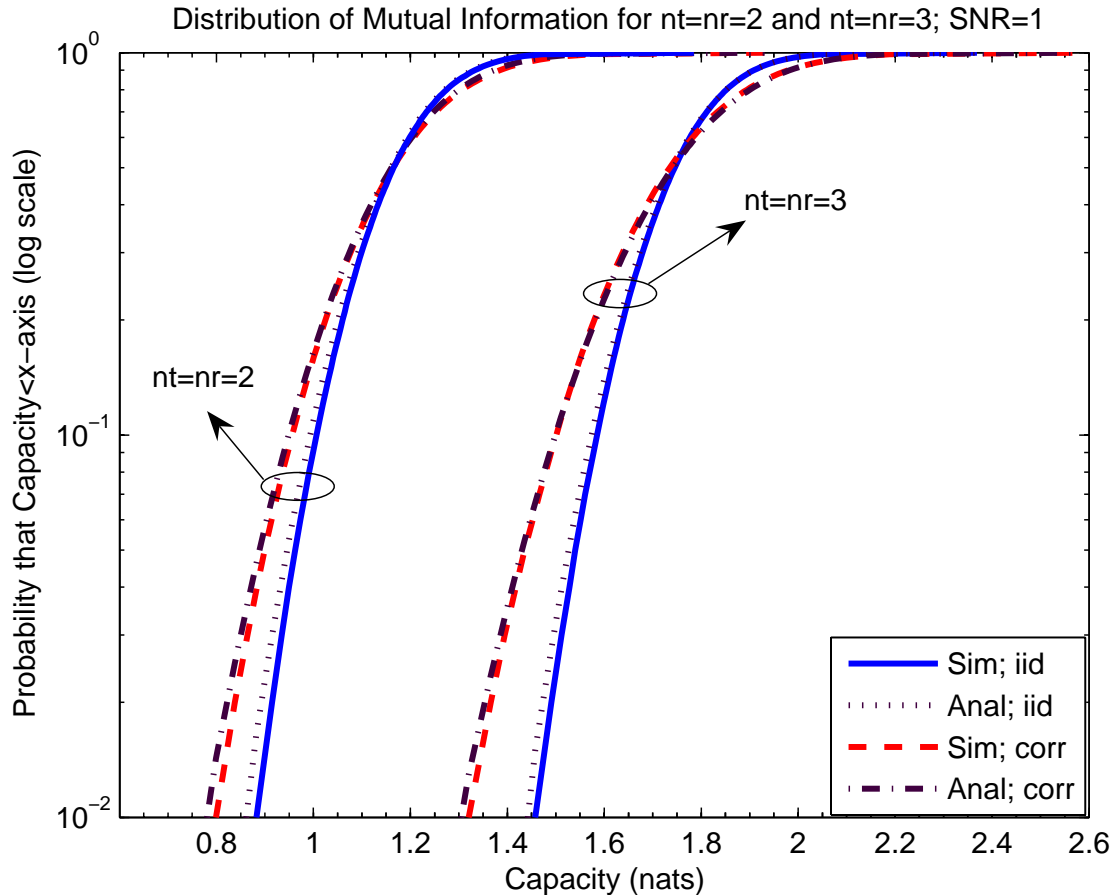


Fig. 2. Cumulative distributions (CDF) of the mutual information with two and three antenna arrays for signal-to-noise ratio (SNR) $\rho = 1$. 10 paths were used, each with an angle-spread of 18 degrees, with the mean angle of arrival of the l -th path pointing at $18(l+1/2)$ degrees. While the mean mutual information is nearly the same for both correlated and iid cases (1.74 nats for $n_t = 3$ and 1.16 nats for $n_t = 2$), the variance of the correlated systems is nearly double the variance of the corresponding iid case (0.357 vs. 0.0171 for $n_t = 3$ and 0.0274 vs. 0.0171 for the $n_t = 2$ case). The agreement between the analytic large N expression and the simulation is very good down to 1% outage.

to

$$r_l = \frac{\rho l q_l}{1 + n_{q_l} \rho l t_l} \quad (71)$$

$$t_l = \frac{1}{1 + n r_l} \quad (72)$$

where q_l are the L eigenvalues of \mathbf{Q} , given by (40). Assuming for simplicity that the ρ_l are ordered, i.e. $\rho_1 \geq \rho_2 \geq \dots \rho_L$, the final solution for the capacity-achieving input distribution covariance matrix is

$$\mathbf{Q} = \frac{1}{n} \sum_{l=1}^{L_{eff}} q_l \mathbf{T}_l \quad (73)$$

$$q_l = \frac{1}{\Lambda_{L_{eff}}} - \frac{1}{\sqrt{\Lambda_{L_{eff}} n \rho_l}} \quad (74)$$

where

$$\Lambda_m = \frac{\left(\sqrt{\alpha_m^2 + 4m} - \alpha_m\right)^2}{4n} \quad (75)$$

$$\alpha_m = \frac{1}{n} \sum_{l=1}^m \frac{1}{\sqrt{\rho_l}}$$

Here, L_{eff} is the number of non-zero \mathbf{Q} eigenvalues, chosen with the condition

$$\Lambda_{L_{eff}} < n \rho_l \quad (76)$$

for all $l \leq L_{eff}$, which comes from the requirement $r_l \geq 0$. The resulting ergodic capacity is

$$I = \sum_{l=1}^{L_{eff}} \left[\log \frac{n \rho_l}{\Lambda_{L_{eff}}} - n \left(1 - \sqrt{\frac{\Lambda_{L_{eff}}}{n \rho_l}} \right) \right] \quad (77)$$

From (35) and (73), we see that the capacity-achieving covariance matrix is a non-trivial linear combination of \mathbf{T}_l , each with coefficient $t_l \rho_l$, which is obtained by solving (71), (72) and (74), which depends on the properties of all paths.

VI. CONCLUSION

In conclusion, we have presented an analytic approach to calculate the statistics of the mutual information of MIMO systems for the most general zero-mean Gaussian wideband channels. We have also shown how the ergodic capacity can be calculated by optimizing over the Gaussian input signal distribution. The analytic approach is in principle valid for large antenna numbers, in which limit the mutual information distribution approaches a Gaussian, irrespective of the wideband richness of the channel. Thus the outage capacity can be explicitly calculated. Nevertheless all results have been found numerically to be valid with high accuracy to arrays with few antennas. Thus our results are applicable to a wide range of multipath problems, including, but not limited to, multipath channels with a few delay taps or to an arbitrary continuous power-delay profile and dual-polarized antennas with arbitrary correlations. It should also be noted that this method generalizes the so-called IND separable channels analyzed in [10] to general non-separable IND channels with arbitrary non-Kronecker product correlations.

This analytic approach provides the framework and a simple tool to accurately analyze the statistics of throughput of even small arrays in the presence of arbitrary channel correlations.

APPENDIX I
COMPLEX INTEGRALS

Identity 1: Let \mathbf{X} , \mathbf{A} , \mathbf{B} be respectively $m \times n$ complex matrices and \mathbf{N} , \mathbf{M} positive-definite hermitian $n \times n$ and $m \times m$. Then, the following equality holds

$$\begin{aligned} & (\det [\mathbf{N} \otimes \mathbf{M}])^{-1} e^{-\frac{1}{2} \text{Tr}\{\mathbf{N}^{-1} \mathbf{A}^\dagger \mathbf{M}^{-1} \mathbf{B}\}} \\ &= \int D\mathbf{X} e^{-\frac{1}{2} \text{Tr}\{\mathbf{N} \mathbf{X}^\dagger \mathbf{M} \mathbf{X} + \mathbf{A}^\dagger \mathbf{X} - \mathbf{X}^\dagger \mathbf{B}\}} \end{aligned} \quad (78)$$

where the integration measure $D\mathbf{X}$ is given by (1).

Proof: See Appendix I in [14]. Note that this formula was printed incorrectly in that reference. Here we state the relevant identity. ■

There are several useful special cases of this identity. Setting, $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$, we obtain

$$\begin{aligned} (\det [\mathbf{N} \otimes \mathbf{M}])^{-1} &= (\det \mathbf{N})^{-m} (\det \mathbf{M})^{-n} \\ &= \int D\mathbf{X} e^{-\frac{1}{2} \text{Tr}\{\mathbf{N} \mathbf{X}^\dagger \mathbf{M} \mathbf{X}\}} \end{aligned} \quad (79)$$

Further setting $\mathbf{N} = \mathbf{I}_n$ yields

$$(\det \mathbf{M})^{-n} = \int D\mathbf{X} e^{-\frac{1}{2} \text{Tr}\{\mathbf{X}^\dagger \mathbf{M} \mathbf{X}\}} \quad (80)$$

Identity 2 (Hubbard-Stratonovich Transformation): Let \mathbf{U} , \mathbf{V} be arbitrary complex $M\nu \times M\nu$ matrices, where ν is assumed to be an arbitrary positive integer. Then the following identity holds

$$e^{-\text{Tr}[\mathbf{U}\mathbf{V}]} = \int d\mu(\mathcal{T}, \mathcal{R}) e^{\text{Tr}[\mathcal{R}\mathcal{T} - \mathbf{U}\mathcal{T} - \mathcal{R}\mathbf{V}]} \quad (81)$$

In the above equation, the auxiliary matrices \mathcal{T} and \mathcal{R} are general complex matrices $M\nu \times M\nu$ and their integration measure is given by (2). The integration of the elements of \mathcal{R} and \mathcal{T} is along contours in complex space parallel to the real and imaginary axis respectively as discussed in [14].

Proof: See Appendix I in [14]. ■

APPENDIX II
DERIVATION OF (28), (29)

In this Appendix we will express $g(\nu)$ as in (28), (29). We start with (19) assuming that ν is an arbitrary positive integer. Using (80) we can write

$$\begin{aligned} & \det(\mathbf{I}_{n_r} + \hat{\mathbf{G}}_k \mathbf{Q} \hat{\mathbf{G}}_k^\dagger)^{-\nu} \\ &= \int D\mathbf{X}_k e^{-\frac{1}{2} \text{Tr}\{\mathbf{x}_k^\dagger \mathbf{x}_k + \mathbf{x}_k^\dagger \hat{\mathbf{G}}_k \mathbf{Q} \hat{\mathbf{G}}_k^\dagger \mathbf{x}_k\}} \end{aligned} \quad (82)$$

where \mathbf{X}_k is an $n_t \times \nu$ -dimensional complex matrix. We then further use (78) to write

$$\begin{aligned} & e^{-\frac{1}{2} \text{Tr}\{\mathbf{x}_k^\dagger \hat{\mathbf{G}}_k \mathbf{Q} \hat{\mathbf{G}}_k^\dagger \mathbf{x}_k\}} \\ &= \int D\mathbf{Y}_k e^{-\frac{1}{2} \text{Tr}\{\mathbf{Y}_k^\dagger \mathbf{Y}_k + \mathbf{x}_k^\dagger \hat{\mathbf{G}}_k \mathbf{Q}^{1/2} \mathbf{Y}_k - \mathbf{Y}_k^\dagger \mathbf{Q}^{1/2} \hat{\mathbf{G}}_k^\dagger \mathbf{x}_k\}} \end{aligned} \quad (83)$$

where \mathbf{Y}_k is also an $n_t \times \nu$ -dimensional complex matrix. Thus, using (82) and (83) and the definition (19) of $g(\nu)$ we can write

$$\begin{aligned} g(\nu) &= \left\langle \prod_k \int D\mathbf{X}_k D\mathbf{Y}_k e^{-\frac{1}{2} \sum_k \text{Tr}\{\mathbf{x}_k^\dagger \mathbf{x}_k + \mathbf{Y}_k^\dagger \mathbf{Y}_k\}} \right. \\ & \quad \left. e^{-\frac{1}{2} \sum_k \text{Tr}\{\mathbf{x}_k^\dagger \hat{\mathbf{G}}_k \mathbf{Q}^{1/2} \mathbf{Y}_k - \mathbf{Y}_k^\dagger \mathbf{Q}^{1/2} \hat{\mathbf{G}}_k^\dagger \mathbf{x}_k\}} \right\rangle \end{aligned} \quad (84)$$

where k ranges from 0 to $M-1$. Note that, as discussed above, we have been able to set all \mathbf{Q}_k equal to a single \mathbf{Q} .

To average the bracketed term over channel realizations we use (9) to express $\hat{\mathbf{G}}$ in terms of \mathbf{G}_l . The probability density of \mathbf{G}_l is defined by (10) and can be rewritten explicitly

$$p(\mathbf{G}_l) = \det \left[\frac{\rho_l}{n_t} \mathbf{T}_l \otimes \mathbf{R}_l \right]^{-1} e^{-\frac{n_t}{2\rho_l} \text{Tr}\{\mathbf{T}_l^{-1} \mathbf{G}_l^\dagger \mathbf{R}_l^{-1} \mathbf{G}_l\}} \quad (85)$$

The expectation bracket of any operator $F(\{\mathbf{G}_l\})$ which is a function of the \mathbf{G}_l 's can then be written as

$$\langle F(\{\mathbf{G}_l\}) \rangle = \prod_{l=0}^{L-1} \int D\mathbf{G}_l p(\mathbf{G}_l) F(\{\mathbf{G}_l\}) \quad (86)$$

Note that using (79) it is easy to see that this probability distribution is properly normalized (i.e., $\langle 1 \rangle = 1$).

We now evaluate the expectation bracket in (84) by rewriting $\hat{\mathbf{G}}$ in terms of \mathbf{G}_l and integrating over the channel realizations (using (85) and (86) and applying (78) to perform the integral). As a result we obtain

$$\begin{aligned} g(\nu) &= \prod_k \int D\mathbf{X}_k D\mathbf{Y}_k e^{-\frac{1}{2} \sum_k \text{Tr}\{\mathbf{x}_k^\dagger \mathbf{x}_k + \mathbf{Y}_k^\dagger \mathbf{Y}_k\}} \\ & \cdot \prod_l e^{-\left[\frac{\rho_l}{2n_t} \sum_{kk'} e^{\frac{2\pi i(k-k')m_l}{M}} \text{Tr}\{\mathbf{x}_{k'}^\dagger \mathbf{R}_l \mathbf{x}_k \mathbf{Y}_k^\dagger \mathbf{Q}^{1/2} \mathbf{T}_l \mathbf{Q}^{1/2} \mathbf{Y}_{k'}\} \right]} \end{aligned} \quad (87)$$

Following [14] we use Identity 2 in Appendix I to express the above in a quadratic form in terms of $\mathbf{X}_k, \mathbf{Y}_k$ by introducing $2L M\nu \times M\nu$ matrices $\mathcal{R}^l, \mathcal{T}^l$. These matrices, whenever convenient, will be represented $\mathcal{R}_{kk'}^l, \mathcal{T}_{kk'}^l$, as a set of LM^2 matrices of dimension $\nu \times \nu$ each. Thus the second line of (87) becomes

$$\begin{aligned} & \int d\mu(\{\mathcal{T}^l, \mathcal{R}^l\}) \prod_l \prod_{kk'} \left(\exp \left[\text{Tr} \left\{ \mathcal{T}_{kk'}^l \mathcal{R}_{kk'}^l \right\} \right] \right. \\ & \cdot \exp \left[-\frac{\rho_l}{2\sqrt{n_t}} e^{\frac{2\pi i(k-k')m_l}{M}} \text{Tr} \left\{ \mathcal{T}_{k'k}^l \mathbf{Y}_k^\dagger \mathbf{Q}^{1/2} \mathbf{T}_l \mathbf{Q}^{1/2} \mathbf{Y}_{k'} \right\} \right] \\ & \cdot \exp \left[-\frac{1}{2\sqrt{n_t}} \text{Tr} \left\{ \mathbf{X}_{k'}^\dagger \mathbf{R}_l \mathbf{X}_k \mathcal{R}_{kk'}^l \right\} \right] \Big) \end{aligned} \quad (88)$$

Combining (87) and (88) and using (79), we can now integrate over $\mathbf{X}_k, \mathbf{Y}_k$, resulting in

$$g(\nu) = \int d\mu(\{\mathcal{T}^l, \mathcal{R}^l\}) e^{-\mathcal{S}} \quad (89)$$

with \mathcal{S} given in (29).

APPENDIX III

DETAILS FOR SADDLE POINT ANALYSIS OF (28), (29)

Using the change of variables $\mathcal{T}^l \rightarrow \delta\mathcal{T}^l, \mathcal{R}^l \rightarrow \delta\mathcal{R}^l$ defined in (31) we expand \mathcal{S} in (29) in powers of $\delta\mathcal{T}^l, \delta\mathcal{R}^l$, resulting in

$$\begin{aligned} \mathcal{S}_0 &= \nu \left[M \log \det \left(\mathbf{I}_{n_t} + \sum_l t_l \rho_l \mathbf{Q} \mathbf{T}_l \right) \right. \\ & \left. + M \log \det \left(\mathbf{I}_{n_r} + \sum_l r_l \mathbf{R}_l \right) - n_t M \sum_l r_l t_l \right] \end{aligned} \quad (90)$$

$$\begin{aligned} \mathcal{S}_1 &= \sum_{k,l} \left[\left(M_{r,1}^l - t_l \sqrt{n_t} \right) \text{Tr} \left\{ \delta\mathcal{R}_{kk}^l \right\} \right. \\ & \left. + \left(M_{t,1}^l - r_l \sqrt{n_t} \right) \text{Tr} \left\{ \delta\mathcal{T}_{kk}^l \right\} \right] \end{aligned} \quad (91)$$

$$\begin{aligned} \mathcal{S}_2 &= -\frac{1}{2} \sum_{kk'} \sum_{ll'} \text{Tr} \left\{ M_{r,2}^{ll'} \delta\mathcal{R}_{kk'}^l \delta\mathcal{R}_{k'k}^{l'} \right. \\ & \left. + M_{t,2}^{ll'} \delta\mathcal{T}_{kk'}^l \delta\mathcal{T}_{k'k}^{l'} + 2\delta\mathcal{T}_{kk'}^l \delta\mathcal{R}_{k'k}^{l'} \right\} \\ &= \frac{1}{2} \sum_{kk'} \text{Tr} \left\{ \mathbf{x}_{kk'} \mathbf{V}^{kk'} \mathbf{x}_{k'k}^T \right\} \end{aligned} \quad (92)$$

where the $2L$ -dimensional vector $\mathbf{x}_{kk'}$ of $\nu \times \nu$ matrices is defined as

$$\mathbf{x}_{kk'} = [\delta\mathcal{R}_{kk'}^0 \dots \delta\mathcal{R}_{kk'}^L \delta\mathcal{T}_{kk'}^0 \dots \delta\mathcal{T}_{kk'}^L] \quad (93)$$

and the corresponding $2L$ -dimensional Hessian $\mathbf{V}^{kk'}$ is expressed in block-diagonal form as

$$\mathbf{V}^{kk'} = \begin{bmatrix} -\mathbf{M}_{r,2} & -\mathbf{I}_L \\ -\mathbf{I}_L & -\mathbf{M}_{t,2} \end{bmatrix} \quad (94)$$

where the matrices $\mathbf{M}_{r,2}$, $\mathbf{M}_{t,2}$ in the diagonals have elements $M_{r,2}^{ll'}$ and $M_{t,2}^{ll'}$, respectively, with $l = 0, \dots, L-1$. For $p > 2$ the expanded terms take the form

$$\begin{aligned} \mathcal{S}_p = & \frac{(-1)^p}{p} \sum_{\mathbf{k}_p, \mathbf{l}_p} \left[M_{r,p}^{l_p} \text{Tr} \left\{ \delta \mathcal{R}_{k_1 k_2}^{l_1} \cdots \delta \mathcal{R}_{k_p k_1}^{l_p} \right\} \right. \\ & \left. + M_{t,p}^{l_p} \text{Tr} \left\{ \delta \mathcal{T}_{k_1 k_2}^{l_1} \cdots \delta \mathcal{T}_{k_p k_1}^{l_p} \right\} \right] \end{aligned} \quad (95)$$

where the p -dimensional integer valued vectors $\mathbf{l} = [l_1 \dots l_p]$, $\mathbf{k}_p = [k_1 \dots k_p]$ are being summed over.

The coefficients in this Taylor expansion have the form

$$M_{t,p}^{l_p} = \text{Tr} \left\{ \prod_{i=1}^p \left[\left(\mathbf{I}_{n_t} + \mathbf{Q} \sum_l t_l \rho_l \mathbf{T}_l \right)^{-1} \frac{\rho_{l_i} \mathbf{Q} \mathbf{T}_{l_i} e^{\frac{2\pi i(k_i - k_{i+1}) m_{l_i}}{M}}}{\sqrt{n_t}} \right] \right\} \quad (96)$$

$$M_{r,p}^{l_p} = \text{Tr} \left\{ \prod_{i=1}^p \left[\left(\mathbf{I}_{n_r} + \sum_l r_l \mathbf{R}_l \right)^{-1} \frac{\mathbf{R}_{l_i}}{\sqrt{n_t}} \right] \right\} \quad (97)$$

Note that while in (88) \mathbf{Q} appears in the form $\mathbf{Q}^{1/2} \mathbf{T}_l \mathbf{Q}^{1/2}$, in (90), (96) it is possible to combine the two $\mathbf{Q}^{1/2}$ into a single \mathbf{Q} .

APPENDIX IV HIGHER ORDER TERMS

In this section we will follow the formulation of [14] to calculate the leading $1/n$ correction to $g(\nu)$ which will contribute as a leading term to the skewness \mathcal{C}_3 and as a correction to the average mutual information \mathcal{C}_1 .

We define an expectation bracket of $F(\delta \mathcal{T}, \delta \mathcal{R})$, an arbitrary function of $\delta \mathcal{T}$, $\delta \mathcal{R}$, as

$$\langle \langle F \rangle \rangle = \prod_{kk'} \left| \det \mathbf{V}^{kk'} \right|^{\nu^2/2} \int d\mu (\delta \mathcal{T}, \delta \mathcal{R}) e^{-\mathcal{S}_2} F(\delta \mathcal{T}, \delta \mathcal{R}) \quad (98)$$

To calculate such expectations, we will expand the function F in its arguments and will then integrate over the Gaussian integral. Thus only integrals over even powers of $\delta \mathcal{T}$, $\delta \mathcal{R}$ will survive. To evaluate

the expanded terms we need the following second order moments (see below)

$$\begin{aligned}
\langle\langle \delta R_{k_1 k_2, ab}^p \delta R_{k_3 k_4, cd}^q \rangle\rangle &= -\delta_{k_1 k_4} \delta_{k_2 k_3} \delta_{ad} \delta_{bc} W_{1, pq}^{k_1 k_2} \\
\langle\langle \delta T_{k_1 k_2, ab}^p \delta T_{k_3 k_4, cd}^q \rangle\rangle &= -\delta_{k_1 k_4} \delta_{k_2 k_3} \delta_{ad} \delta_{bc} W_{2, pq}^{k_1 k_2} \\
\langle\langle \delta T_{k_1 k_2, ab}^p \delta R_{k_3 k_4, cd}^q \rangle\rangle &= -\delta_{k_1 k_4} \delta_{k_2 k_3} \delta_{ad} \delta_{bc} W_{3, pq}^{k_1 k_2}
\end{aligned} \tag{99}$$

where for each $k_1, k_2 = 1, \dots, \nu$, the $L \times L$ matrices $\mathbf{W}_i^{k_1 k_2}$ for $i = 1, \dots, 3$ are given in terms of the $L \times L$ matrices $\mathbf{M}_{r,2}, \mathbf{M}_{t,2}$ (see (96), (97)) by the following expressions

$$\begin{aligned}
\mathbf{W}_1^{k_1 k_2} &= -\mathbf{M}_{t,2} [\mathbf{M}_{r,2} \mathbf{M}_{t,2} - \mathbf{I}_L]^{-1} \\
\mathbf{W}_2^{k_1 k_2} &= -\mathbf{M}_{r,2} [\mathbf{M}_{t,2} \mathbf{M}_{r,2} - \mathbf{I}_L]^{-1} \\
\mathbf{W}_3^{k_1 k_2} &= [\mathbf{M}_{r,2} \mathbf{M}_{t,2} - \mathbf{I}_L]^{-1}
\end{aligned} \tag{100}$$

independent of k_1, k_2 . In our particular case, the function $F(\delta\mathcal{T}, \delta\mathcal{R})$ is $\exp[-\sum_{p>2} \mathcal{S}_p]$, with \mathcal{S}_p expressed in terms of $\delta\mathcal{T}, \delta\mathcal{R}$, as in (95). We now expand the exponential by combining terms with equal powers of n . To do this, we note in (95) that $\langle\langle \mathcal{S}_p \rangle\rangle$ is of order $n^{-p/2+1}$ for p even, while it is zero for p odd. Keeping only the $\mathcal{O}(n^{-1})$ terms, $g(\nu)$ takes the form

$$g(\nu) = e^{-\nu M \Gamma} \prod_{pq} |\det \mathbf{V}^{pq}|^{-\nu^2/2} [1 + D_1 + \mathcal{O}(n^{-2})] \tag{101}$$

where

$$D_1 = \langle\langle \mathcal{S}_4 + \frac{1}{2} \mathcal{S}_3^2 \rangle\rangle \tag{102}$$

which is of order $1/n$.

To evaluate D_1 we need to calculate $\langle\langle \mathcal{S}_4 \rangle\rangle$ and $\langle\langle \mathcal{S}_3^2 \rangle\rangle$, which, as seen in (95), include fourth order and sixth order products in $\delta\mathcal{T}, \delta\mathcal{R}$, respectively. These can be calculated by applying Wick's theorem (see [14] or [50]), i.e. by "pairing" all $\delta\mathcal{T}$'s and $\delta\mathcal{R}$'s with each other and using (99) to calculate the corresponding quadratic moments. As an example, we evaluate below the term in \mathcal{S}_4 which is proportional

to $M_{r,4}^{p_1 p_2 p_3 p_4}$ in (95).

$$\begin{aligned}
& \sum_{k_1, \dots, k_4=1}^M \sum_{a, b, c, d=1}^{\nu} \langle \langle \delta R_{k_1 k_2, ab}^{p_1} \cdots \delta R_{k_4 k_1, da}^{p_4} \rangle \rangle \quad (103) \\
&= \sum_{k_1, \dots, k_4=1}^M \sum_{a, b, c, d=1}^{\nu} [\\
& \quad \langle \langle \delta R_{k_1 k_2, ab}^{p_1} \delta R_{k_2 k_3, bc}^{p_2} \rangle \rangle \langle \langle \delta R_{k_3 k_4, cd}^{p_3} \delta R_{k_4 k_1, da}^{p_4} \rangle \rangle \\
& \quad + \langle \langle \delta R_{k_1 k_2, ab}^{p_1} \delta R_{k_3 k_4, cd}^{p_3} \rangle \rangle \langle \langle \delta R_{k_2 k_3, bc}^{p_2} \delta R_{k_4 k_1, da}^{p_4} \rangle \rangle \\
& \quad + \langle \langle \delta R_{k_1 k_2, ab}^{p_1} \delta R_{k_4 k_1, da}^{p_4} \rangle \rangle \langle \langle \delta R_{k_2 k_3, bc}^{p_2} \delta R_{k_3 k_4, cd}^{p_3} \rangle \rangle] \\
&= \nu^3 \left(W_{1, p_1 p_3}^{k_1 k_2} W_{1, p_3 p_4}^{k_1 k_3} + W_{1, p_1 p_4}^{k_1 k_2} W_{1, p_2 p_3}^{k_2 k_3} \right) \\
& \quad + \nu M W_{1, p_1 p_3}^{k_1 k_1} W_{1, p_2 p_4}^{k_1 k_1} \quad (104)
\end{aligned}$$

We can similarly evaluate the second term in \mathcal{S}_4 as well as \mathcal{S}_3^2 to get

$$D_1 = a_1 \nu + a_3 \nu^3 \quad (105)$$

where

$$\begin{aligned}
a_1 &= \sum_{p_1 \dots p_4} \left\{ \frac{M}{4} \left[M_{r,4}^{p_1 p_2 p_3 p_4} W_{1, p_1 p_3}^{kk} W_{1, p_2 p_4}^{kk} \right. \right. \quad (106) \\
& \quad \left. \left. + M_{t,4}^{p_1 p_2 p_3 p_4} W_{2, p_1 p_3}^{kk} W_{2, p_2 p_4}^{kk} \right] \right. \\
& \quad + \sum_{p_1 \dots p_6} \left\{ \frac{M}{6} \left[M_{r,3}^{p_1 p_2 p_3} M_{r,3}^{p_4 p_5 p_6} W_{1, p_1 p_2}^{kk} W_{1, p_3 p_4}^{kk} W_{1, p_5 p_6}^{kk} \right. \right. \\
& \quad \left. \left. + M_{t,3}^{p_1 p_2 p_3} M_{t,3}^{p_4 p_5 p_6} W_{2, p_1 p_2}^{kk} W_{2, p_3 p_4}^{kk} W_{2, p_5 p_6}^{kk} \right. \right. \\
& \quad \left. \left. + 2M_{r,3}^{p_1 p_2 p_3} M_{t,3}^{p_4 p_5 p_6} W_{3, p_1 p_4}^{kk} W_{3, p_2 p_5}^{kk} W_{3, p_3 p_6}^{kk} \right] \right\}
\end{aligned}$$

and

$$\begin{aligned}
a_3 = & \frac{1}{4} \sum_{p_1 \dots p_4} \sum_{k_1 k_2 k_3} \{ \\
& M_{r,4}^{p_1 p_2 p_3 p_4} \left[W_{1,p_1 p_2}^{k_1 k_2} W_{1,p_3 p_4}^{k_1 k_3} + W_{1,p_1 p_4}^{k_1 k_2} W_{1,p_2 p_3}^{k_2 k_3} \right] \\
& + M_{t,4}^{p_1 p_2 p_3 p_4} \left[W_{2,p_1 p_2}^{k_1 k_2} W_{2,p_3 p_4}^{k_1 k_3} + W_{2,p_1 p_4}^{k_1 k_2} W_{2,p_2 p_3}^{k_2 k_3} \right] \} \\
+ & \frac{1}{6} \sum_{p_1 \dots p_6} \sum_{k_1 k_2 k_3} \left\{ M_{r,3}^{p_1 p_2 p_3} M_{r,3}^{p_4 p_5 p_6} \right. \\
& \left[3W_{1,p_1 p_2}^{k_1 k_2} W_{1,p_3 p_4}^{k_1 k_1} W_{1,p_5 p_6}^{k_1 k_3} + W_{1,p_1 p_4}^{k_1 k_2} W_{1,p_2 p_6}^{k_2 k_3} W_{1,p_3 p_5}^{k_3 k_1} \right] \\
& + M_{t,3}^{p_1 p_2 p_3} M_{t,3}^{p_4 p_5 p_6} \\
& \left[3W_{2,p_1 p_2}^{k_1 k_2} W_{2,p_3 p_4}^{k_1 k_1} W_{2,p_5 p_6}^{k_1 k_3} + W_{2,p_1 p_4}^{k_1 k_2} W_{2,p_2 p_6}^{k_2 k_3} W_{2,p_3 p_5}^{k_3 k_1} \right] \\
& + 2M_{r,3}^{p_1 p_2 p_3} M_{t,3}^{p_4 p_5 p_6} \\
& \left. \left[3W_{1,p_1 p_2}^{k_1 k_2} W_{2,p_5 p_6}^{k_1 k_3} W_{3,p_3 p_4}^{k_1 k_1} + W_{3,p_1 p_4}^{k_1 k_2} W_{3,p_2 p_6}^{k_2 k_3} W_{3,p_3 p_2}^{k_3 k_1} \right] \right\}
\end{aligned} \tag{107}$$

APPENDIX V

CAPACITY-ACHIEVING INPUT SIGNAL COVARIANCE \mathbf{Q}

In this Appendix we will show that the capacity-achieving input distribution \mathbf{Q} is diagonal in the basis of $\tilde{\mathbf{T}}$ defined in (35). To start the proof we point out that the mutual information (to order $\mathcal{O}(1/n)$) for a given \mathbf{Q} is the extremum of (38) given also below

$$\begin{aligned}
\Gamma(\mathbf{Q}, \{t_l\}, \{r_l\}) = & M \log \det \left(\mathbf{I}_{n_t} + \mathbf{Q} \tilde{\mathbf{T}} \right) \\
& + M \log \det \left(\mathbf{I}_{n_r} + \tilde{\mathbf{R}} \right) - n_t M \sum_l r_l t_l
\end{aligned}$$

with respect to t_l, r_l for $l = 0, \dots, L-1$. The saddle-point equations are given by (33), (34) also seen below:

$$\begin{aligned}
r_l = & \frac{\rho_l}{n_t} \text{Tr} \left\{ \mathbf{Q} \mathbf{T}_l \left[\mathbf{I}_{n_t} + \mathbf{Q} \tilde{\mathbf{T}} \right]^{-1} \right\} \\
t_l = & \frac{1}{n_t} \text{Tr} \left\{ \mathbf{R}_l \left[\mathbf{I}_{n_r} + \tilde{\mathbf{R}} \right]^{-1} \right\}
\end{aligned}$$

It should be noted that the mutual information is an extremum of \mathcal{S} in a larger complex space of the elements of the matrices $\{\mathcal{T}^l, \mathcal{R}^l\}$, but for simplicity we only focus on the dependence of $\Gamma(\mathbf{Q}, \{t_l\}, \{r_l\})$ in the $2L$ -dimensional space of $\{t_l, r_l\}$. In this case one can view $\Gamma(\mathbf{Q}, \{t_l\}, \{r_l\})$ as a function of

$2L + n_t^2 + n_t - 1$ variables, where the last $n_t^2 + n_t - 1$ are the degrees of freedom of \mathbf{Q} , an n_t -dimensional hermitian complex matrix with fixed trace. Extremizing the above function over $\{t_l, r_l\}$ we can eliminate all $\{t_l, r_l\}$ using the above equations. Thus for fixed \mathbf{Q} the mutual information can be written as $I(\mathbf{Q}) = \Gamma(\mathbf{Q}, \{t_l(\mathbf{Q})\}, \{r_l(\mathbf{Q})\})$ with $t_l(\mathbf{Q}), r_l(\mathbf{Q})$ functions of \mathbf{Q} . Suppose now that we maximize $I(\mathbf{Q})$ with respect to \mathbf{Q} with the constraint $\text{Tr}\mathbf{Q} = n_t$ and that \mathbf{Q}_0 is the optimal matrix. As a result, $I(\mathbf{Q}_0) = \Gamma(\mathbf{Q}_0, \{t_l(\mathbf{Q}_0)\}, \{r_l(\mathbf{Q}_0)\})$ is a maximum over \mathbf{Q} and an extremum over t_l, r_l . Thus if one varies \mathbf{Q}_0 locally keeping its eigenvalues (and trace) fixed, the variation of $I(\mathbf{Q})$ will vanish to first order in the variation. The most general such variation can be written as

$$\mathbf{Q}_\lambda = e^{i\lambda\mathbf{H}}\mathbf{Q}_0e^{-i\lambda\mathbf{H}} \quad (108)$$

$$= \mathbf{Q}_0 + i\lambda[\mathbf{H}, \mathbf{Q}_0] + \dots \quad (109)$$

where $\mathbf{H} = \mathbf{H}^\dagger$ is an arbitrary traceless Hermitian matrix, λ is a small scalar, and the notation $[a, b] = ab - ba$ is the commutator. Thus the first derivative of Γ with λ has to vanish at $\lambda = 0$. Therefore we have

$$0 = \left. \frac{d\Gamma}{d\lambda} \right|_{\lambda=0} = \left[\frac{\partial\Gamma}{\partial\lambda} + \sum_l \left\{ \frac{\partial\Gamma}{\partial t_l} \frac{dt_l}{d\lambda} + \frac{\partial\Gamma}{\partial r_l} \frac{dr_l}{d\lambda} \right\} \right] \Big|_{\lambda=0} \quad (110)$$

Since Γ is an extremum with respect to $\{r_l, t_l\}$ the partial derivatives of Γ with $\{r_l, t_l\}$ vanish. We are left with the first term, $\partial\Gamma/\partial\lambda$, which should also vanish if Γ is a maximum over \mathbf{Q} , resulting to

$$0 = \left. \frac{\partial\Gamma}{\partial\lambda} \right|_{\lambda=0} = \text{Tr} \left[[\mathbf{H}, \mathbf{Q}_0] \tilde{\mathbf{T}} \left(\mathbf{I}_{n_t} + \mathbf{Q}_0 \tilde{\mathbf{T}} \right)^{-1} \right] \quad (111)$$

$$= \text{Tr}[\mathbf{HZ}] \quad (112)$$

with

$$\mathbf{Z} = \left(\mathbf{I}_{n_t} + \mathbf{Q}_0 \tilde{\mathbf{T}} \right)^{-1} - \left(\mathbf{I}_{n_t} + \tilde{\mathbf{T}} \mathbf{Q}_0 \right)^{-1} \quad (113)$$

Now, since \mathbf{H} is an arbitrary traceless Hermitian matrix, the condition (112) is equivalent to the statement that \mathbf{Z} is proportional to the identity matrix. However, it is easy to see from (113) that \mathbf{Z} must be traceless, which implies that our extremization condition is equivalent to $\mathbf{Z} = \mathbf{0}$ or

$$\tilde{\mathbf{T}}\mathbf{Q}_0 = \mathbf{Q}_0\tilde{\mathbf{T}} \quad (114)$$

which requires that \mathbf{Q}_0 and $\tilde{\mathbf{T}}$ have the same eigenvectors whenever \mathbf{Q}_0 is a maximum.

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