

Capacity and Character Expansions: Moment generating function and other exact results for MIMO correlated channels

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Abstract

We apply a promising new method from the field of representations of Lie groups to calculate integrals over unitary groups, which are important for multi-antenna communications. To demonstrate the power and simplicity of this technique, we first re-derive a number of results that have been used recently in the community of wireless information theory, using only a few simple steps. In particular, we derive the joint probability distribution of eigenvalues of the matrix $\mathbf{G}\mathbf{G}^\dagger$, with \mathbf{G} a semicorrelated Gaussian random matrix or a Gaussian random matrix with a non-zero mean. These joint probability distribution functions can then be used to calculate the moment generating function of the mutual information for Gaussian MIMO channels with these probability distribution of their channel matrices \mathbf{G} . We then turn to the previously unsolved problem of calculating the moment generating function of the mutual information of MIMO channels, which are correlated at both the receiver and the transmitter. From this moment generating function we obtain the ergodic average of the mutual information and study the outage probability. These methods can be applied to a number of other problems. As a particular example, we examine unitary encoded space-time transmission of MIMO systems and we derive the received signal distribution when the channel matrix is correlated at the transmitter end.

Index Terms

Capacity, Space-Time Coding, Multiple Antennas, Unitary Space-Time coding, MIMO, Random Matrix Theory, Character expansions.

I. INTRODUCTION

In recent years, a flurry of research has resulted from the prediction [1], [2] that the use of multiple antennas to transmit and receive signals leads to substantial increases in information throughput. In an effort to quantify these throughput gains, several authors have analyzed the asymptotic behavior of the multiple-input-multiple-output (MIMO) mutual information for large antenna numbers. In [3], [4] the asymptotic ergodic capacity (mutual information averaged over channel realizations) was evaluated in closed form for Gaussian i.i.d. channels. In [5]–[8] the assumption of i.i.d. channels was lifted and the effects of spatial correlations were included. In addition, asymptotic results on the variance of the mutual information were derived, [9]–[12], which together with its mean were shown [9] to describe the full distribution very accurately even for small antenna numbers.

More recently, several exact results regarding the distribution of mutual information for the MIMO link have appeared in the literature. Using different methods these prior works calculate the moment generating function of the distribution of the mutual information for various assumptions about the statistics of the

channel matrix. From the generating function, the moments can be generated by direct differentiation, or the probability of outage [13] can be derived through a simple numerical integral. Specifically, [14] first extracted the moment-generating function for Gaussian uncorrelated channels. Furthermore, in [15], [16] Rician channels were used. In addition, [17], [18] treated the case of semi-correlated channels, where either the transmitting or receiving antennas are correlated (but not both). In particular, [17] dealt with the case where the number of correlated antennas is less than or equal to the number of uncorrelated antennas, while [18] dealt with the opposite case. All above progress in the calculation of the moment generating function relied essentially on previously known results in the theory of Wishart matrices (matrices of the form $\mathbf{G}\mathbf{G}^\dagger$), which allowed the calculation of the joint probability distribution of eigenvalues of such matrices.

Interestingly, one can formulate the problem of calculating the joint probability distribution of eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ in terms of integrals over the unitary group. Such integrals have been analyzed since the 1980's in the field of high energy physics [19], [20] and even earlier in the mathematics community [21]. More recently, [22] recast the same problem from an elegant group-theoretic point of view, which allows one to perform such integrals in a few simple steps. In this paper we will apply these new group-theoretic methods to calculate the joint distribution of eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$, for a number of different channel distributions, $p(\mathbf{G})$. From the eigenvalue distribution the moment generating function of the mutual information is readily obtained. We first simply re-derive a few of the above mentioned prior results [15], [17], [18] and then we obtain a new result of the moment generating function for zero-mean Gaussian channels with correlations in both (transmitting and receiving) ends of the link.¹ In addition, we discuss in an appendix how similar techniques can be powerful in analyzing unitary space-time encoded transmission in MIMO systems. [24]–[27].

The techniques we use are both simple and powerful, and a main aim of this paper is to expose the reader to these methods. Although we cannot attempt to give a complete exposition of the field of group-theory, we are able to give a small and simple set of rules, definitions, and theorems that will allow the interested reader to apply these techniques to other problems of interest.

A. Outline

We start with some notational definitions and the introduction of the system model in Section II. In Section III we introduce the reader to some basic definitions in group theory and quote the basic results

¹In [23] we derive a similar result used to calculate the marginal eigenvalue distribution of $\mathbf{G}\mathbf{G}^\dagger$ with such statistics.

of the method, which will be applied in the remainder of the paper. We illustrate the main steps of this method by calculating the known joint probability distribution of the eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ for the semicorrelated Gaussian MIMO channel \mathbf{G} in Section IV-B, which involves a single unitary integral. In Section IV-C we generalize our method to derive the joint distribution of eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ for \mathbf{G} being a non-zero mean Gaussian MIMO channel. Although also a known result, this analysis gives us further insight in the utility of our method, since this case requires a double unitary integration.

In Sections IV-D and V we apply the technique to the case the zero-mean Gaussian channel with correlated transmitting and receiving antennas and derive a closed-form expression for the moment generating function for the corresponding mutual information, which is a new result. We use this formula in Section VI to obtain an analytical expression for the ergodic capacity and also study the outage probability.

As mentioned above, in this paper we aim to expose the reader to the new group-theoretic methods discussed in Section III. These methods are also applicable in other areas of information theory of MIMO systems. To illustrate this point, we apply this method in the context of unitary space-time coding and calculate the received signal distribution for the case where the channel is correlated at the transmitter end, a new result. In order not to disrupt the flow of the main part, this analysis appears in Appendix V.

II. DEFINITIONS

A. Notation

1) *Vectors/Matrices*: Throughout this paper we will use bold-faced upper-case letters to denote matrices, e.g. \mathbf{X} , with elements given by X_{ab} . Bold-faced lower-case letters, e.g. \mathbf{x} will be used for vectors with elements x_i and non-bold lower-case letters will denote scalar quantities. Also, the superscripts T and \dagger will indicate transpose and Hermitian conjugate operations and \mathbf{I}_n will represent the n -dimensional identity matrix. Finally, $diag(\mathbf{x})$ will denote a diagonal, but not necessarily square matrix with the elements of the vector \mathbf{x} on the main diagonal.

2) *Determinants*: $\det(f(i, j))$ will denote the determinant of a matrix with the i, j th element given by $f(i, j)$, an arbitrary function of i and j . In order to keep the notation consistent, we will always use the indices i, j for this purpose in determinants. Furthermore, we will be using the notation $\Delta(\mathbf{x})$ for an n -dimensional Vandermonde determinant of the elements of the n -dimensional vector \mathbf{x} .

$$\Delta(\mathbf{x}) = \Delta([x_1, \dots, x_n]) = \det(x_i^{j-1}) = \prod_{i>j} (x_i - x_j) \quad (1)$$

In addition, due to the extensive use of determinant expansion in the text, it is convenient to define the following notation

$$\det \mathbf{X} = \sum_{\mathbf{a}} \text{sgn}(\mathbf{a}) \prod_{i=1}^M X_{ia_i} = \frac{1}{M!} \sum_{\mathbf{a}} \sum_{\mathbf{b}} \text{sgn}(\mathbf{a}) \text{sgn}(\mathbf{b}) \prod_{i=1}^M X_{a_i b_i} \quad (2)$$

where \mathbf{X} is a M -dimensional matrix. In the above expression the vector $\mathbf{a} = [a_1, a_2, \dots, a_M]$ is a permutation of the sequence $[1, 2, \dots, M]$. $\text{sgn}(\mathbf{a})$ is the sign of the permutation a , which is $\text{sgn}(\mathbf{a}) = +1$ if $[a_1, a_2, \dots, a_M]$ is an even permutation of the sequence $[1, 2, \dots, M]$, $\text{sgn}(\mathbf{a}) = -1$ for an odd permutation. It is implied here that if \mathbf{a} is not a permutation of $[1, 2, \dots, M]$, $\text{sgn}(\mathbf{a}) = 0$. The summation is over $1 \leq a_i \leq M$ for $i = 1, \dots, M$.

3) *Integral Measures*: We will use three types of integrals: In the first we will be integrating over the real and imaginary part of the elements of a complex $m_{rows} \times m_{cols}$ matrix \mathbf{X} . The integral measure will be denoted by

$$\mathcal{D}\mathbf{X} = \prod_{a=1}^{m_{rows}} \prod_{\alpha=1}^{m_{cols}} \frac{d\text{Re}X_{a\alpha} d\text{Im}X_{a\alpha}}{\pi} \quad (3)$$

We will also be using the notation

$$d\mathbf{x} = \prod_{i=1}^n dx_i \quad (4)$$

for integrals over the positive real axis of the elements of the vector \mathbf{x} . Finally, DU will represent the Haar integral measure [28] for integrals over the elements of the unitary group $U(M)$ for a specified M .

4) *Antenna Numbers*: n_t and n_r will denote the number of antennas at the transmitter and receiver, respectively. We will also use the following notation for maxima and minima of antennas: $M = \max(n_t, n_r)$, $N = \min(n_r, n_t)$.

B. System Model

We consider the case of single-user transmission from n_t transmit antennas to n_r receive antennas over a narrow band fading channel. The received n_r -dimensional complex signal vector \mathbf{y} can be written as

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{z} \quad (5)$$

\mathbf{G} is a $n_r \times n_t$ complex matrix of the channel coefficients from the transmitting to the receiving arrays. \mathbf{x} is the n_t -dimensional vector of transmitted Gaussian signals, while \mathbf{z} is the n_r -dimensional additive Gaussian noise vector. Without loss of generality, both \mathbf{x} and \mathbf{z} are taken to be i.i.d., zero-mean with unit-variance. One can straightforwardly include arbitrary covariances for both in the end result (e.g. see [9]). Finally, the channel \mathbf{G} is assumed to be block-fading, i.e. constant over a significant amount of time

and then changing completely according to its statistics. For concreteness, Gaussian channel statistics of \mathbf{G} are assumed.

During reception, the receiver is assumed to know the instantaneous channel matrix \mathbf{G} . Thus, the mutual information can be expressed as [1]

$$I(\mathbf{y}; \mathbf{x} | \mathbf{G}) = \log \det (\mathbf{I}_{n_t} + \mathbf{G}^\dagger \mathbf{G}) \quad (6)$$

The log above (and throughout the whole paper) represents the natural logarithm and thus I is expressed in nats.

We describe the statistics of the channel \mathbf{G} in terms of its probability distribution $p(\mathbf{G})$. Due to the underlying randomness of \mathbf{G} , the mutual information I is also a random quantity. In this paper we will analyze the statistics of the mutual information I in (6), assuming that the channel \mathbf{G} is a Gaussian random matrix. We will specify details of several different types of Gaussian distributions to study below (iid, semicorrelated, nonzero mean, and correlated). In order to analyze the statistics of the mutual information I , we will calculate the generating function $g(z)$ defined as,

$$g(z) = E_{\mathbf{G}} [e^{zI}] = E_{\mathbf{G}} \left[\left\{ \det (\mathbf{I}_{n_t} + \mathbf{G}^\dagger \mathbf{G}) \right\}^z \right] \quad (7)$$

where the expectation $E_{\mathbf{G}}[\cdot]$ is with respect to the probability distribution $p(\mathbf{G})$ of the matrix \mathbf{G} . Once $g(z)$ is known, all moments of I can be generated by evaluating the derivatives of $g(z)$ at $z = 0$.

$$g'(z)|_{z=0} = E_{\mathbf{G}} [I] \quad (8)$$

$$g''(z)|_{z=0} = E_{\mathbf{G}} [I^2] \quad (9)$$

etc. In addition, the probability of outage, P_{out} , i.e. the probability that the mutual information is less than a given value I_{out} [13], can be readily obtained by performing the integral

$$P_{out} = E_{\mathbf{G}} [\Theta(I - I_{out})] = \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} g(iz) \frac{e^{-izI_{out}}}{z - i0^+} \quad (10)$$

where we have used the identity $\Theta(x) = \int dz/(2\pi i) \exp(izx)/(z - i0^+)$ for the Heavyside step function $\Theta(x)$.

From the form of the mutual information in (6) and (7), it is clear that the generating function can be written very simply in terms of the eigenvalues λ_i of the matrix $\mathbf{G}^\dagger \mathbf{G}$, as

$$g(z) = E_{\boldsymbol{\lambda}} \left[\prod_{i=1}^N (1 + \lambda_i)^z \right] = \prod_{i=1}^N \int_0^\infty d\lambda_i (1 + \lambda_i)^z P(\{\lambda_j\}) \quad (11)$$

where the product is over the $N = \min(n_t, n_r)$ non-zero eigenvalues of $\mathbf{G}^\dagger \mathbf{G}$ and the expectation $E_{\boldsymbol{\lambda}}[\cdot]$ is defined as a multiple integral over the joint distribution $P(\{\lambda_i\})$ of the eigenvalues $\{\lambda_i\}$. Note that $P(\{\lambda_i\})$ will be different for each distribution $p(\mathbf{G})$ we will analyze.

III. IMPORTANT RESULTS FROM REPRESENTATION THEORY OF UNITARY GROUPS

In this section we will briefly introduce a few relevant quantities and a number of important results from the theory of representations of unitary groups, which will be used in the paper. The interested reader can refer to several textbooks, including [28], [29]. We do not attempt a thorough exposition of the subject of group theory. However, we do aim to give a self-contained introduction, so that the interested reader can effectively apply these methods to tackle other problems of interest.

A d -dimensional representation V of a group G is a homomorphism from G into a group of d -dimensional invertible matrices. Such groups include $Gl(M)$, the group of complex invertible matrices of dimension M , and $U(M)$ is its subgroup of unitary matrices (i.e., the subgroup of matrices \mathbf{U} such that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$). These groups are also called Lie groups, i.e. differentiable manifolds with both multiplication and inverse operations being differentiable.

1) *Irreducible Representations:* A d -dimensional representation is *irreducible* if it has no non-trivial invariant subspaces. In other words, a representation is irreducible if there exists no d -dimensional invertible matrix \mathbf{A} , such that $\mathbf{A}\mathbf{H}\mathbf{A}^{-1}$ becomes block diagonal for all group elements \mathbf{H} .

The irreducible representations of the unitary group $U(M)$ (and $Gl(M)$) can both be labelled by an M -dimensional vector $\mathbf{m} = [m_1, m_2, \dots, m_M]$, with integers $m_1 \geq m_2 \geq \dots \geq m_M \geq 0$. [28], [29]

2) *Dimension of Irreducible Representations:* The dimension $d_{\mathbf{m}}$ of an irreducible representation labelled by \mathbf{m} is the dimension of its invariant subspace. For both cases of $U(M)$ and $Gl(M)$ it can be shown [30] that $d_{\mathbf{m}}$ is given by

$$d_{\mathbf{m}} = \left[\prod_{i=1}^M \frac{(M + m_i - i)!}{(M - i)!} \right] \det \left[\frac{1}{(m_i - i + j)!} \right] \quad (12)$$

where the matrix elements inside the determinant with $m_i - i + j < 0$ are zero.

The above form of the dimension $d_{\mathbf{m}}$ in terms of a determinant of a matrix of inverse factorials is not particularly useful for manipulation. The following lemma makes its form more manageable. Its proof appears in Appendix I.

Lemma 1 (Vandermonde determinant form of $d_{\mathbf{m}}$): The dimension $d_{\mathbf{m}}$ of an irreducible representation \mathbf{m} of $U(M)$, given in (12) can be written as

$$d_{\mathbf{m}} = \left[\prod_{i=1}^M \frac{1}{(M - i)!} \right] (-1)^{\frac{M(M-1)}{2}} \Delta(\mathbf{k}) \quad (13)$$

where $\Delta(\cdot)$ represents the Vandermonde determinant and the vector \mathbf{k} has elements

$$k_i = m_i - i + M \quad (14)$$

for $i = 1, \dots, M$ and m_i are the elements of the representation vector \mathbf{m} .

3) *Character of Representation:* The character $\chi(g)$ of a group element g in the representation V is equal to the trace of the corresponding matrix, i.e. $\chi(g) = \text{Tr}\{V(g)\}$. Also, $\chi(g)$ depends only on the eigenvalues of $V(g)$. Calculating the characters of irreducible representations is greatly facilitated by Weyl's character formula [29], [31], which for $U(M)$ and $Gl(M)$ takes the form:

$$\chi_{\mathbf{m}}(\mathbf{A}) = \text{Tr}\{\mathbf{A}^{(\mathbf{m})}\} = \frac{\det\left(a_i^{m_j+M-j}\right)}{\Delta(a_1, \dots, a_M)} \quad (15)$$

where the index \mathbf{m} denotes the irreducible representation (m_1, \dots, m_M) and a_i , for $i = 1, \dots, M$, are the eigenvalues of \mathbf{A} in the fundamental (M -dimensional) representation. Here \mathbf{A} in $\chi_{\mathbf{m}}(\mathbf{A})$ represents a M -dimensional matrix corresponding to the group element that is being represented, while $\mathbf{A}^{(\mathbf{m})}$ represents the $d_{\mathbf{m}}$ -dimensional matrix, which is the (\mathbf{m}) -representation of \mathbf{A} .

For example, the characters of the one-dimensional unitary group $U(1)$ are given by $\chi_n(e^{i\phi}) = e^{in\phi}$, where the character index n takes non-negative integer values and $e^{i\phi}$ is an arbitrary element in $U(1)$. Thus a Fourier transform² can be seen as an expansion in the characters of $U(1)$. This suggests that the characters of a group can form a good basis for expanding functions, which are invariant under the corresponding group operations. Although such expansions are generally complicated, recently [22] showed how it can be done for exponentials of a trace of a $Gl(M)$ matrix. His result can be summarized in the following important lemma.

Lemma 2 (Character Expansion of Exponential): Let \mathbf{A} be a member of the general linear group $Gl(M)$ and x an arbitrary constant. Then the following equation holds:

$$e^{x\text{Tr}\{\mathbf{A}\}} = \sum_{\mathbf{m}} \alpha_{\mathbf{m}}(x) \chi_{\mathbf{m}}(\mathbf{A}) \quad (16)$$

In the above expression, $\chi_{\mathbf{m}}(\mathbf{A})$ is the character of \mathbf{A} in the representation \mathbf{m} and the sum is over all irreducible representations of $Gl(M)$ parameterized with the vector $\mathbf{m} = [m_1, m_2, \dots, m_M]$, with integers $m_1 \geq m_2 \geq \dots \geq m_M \geq 0$. In (16) the coefficient for each character, $\alpha_{\mathbf{m}}(x)$ is given by

$$\alpha_{\mathbf{m}}(x) = (x)^{m_1+m_2+\dots+m_M} \left[\prod_{i=1}^M \frac{(M-i)!}{(M+m_i-i)!} \right] d_{\mathbf{m}} \quad (17)$$

where $d_{\mathbf{m}}$ is the dimension of the irreducible representation and is given by (12).

The importance of this result is that it allows us to expand the exponential in group-invariant quantities as the characters $\chi_{\mathbf{m}}$.

²The astute reader will realize that the characters have only nonnegative n , whereas the Fourier transform has all n . This means the Fourier transform is actually a character expansion of U and U^\dagger . This complication does not enter into our particular use of the character expansion.

4) *Orthogonality Relation between Unitary Group Matrix Elements:* The integration properties of elements of the unitary group in different representations are given in the following (see [28])

$$\int D\mathbf{U} U_{ij}^{(\mathbf{m})} U_{kl}^{(\mathbf{m}')*} = \frac{1}{d_{\mathbf{m}}} \delta_{\mathbf{m}\mathbf{m}'} \delta_{ik} \delta_{jl} \quad (18)$$

where \mathbf{m} is the dimension of the representation and $D\mathbf{U}$ is the standard Haar integration measure over $U(M)$.

5) *Cauchy-Binet Formula:* Finally, we cite a useful result, which will be applied in our calculations below (see proof in [29]).

Lemma 3 (Cauchy-Binet Formula): Given M -dimensional vectors \mathbf{a} and \mathbf{b} , and a function $W(z) = \sum_{i=0}^{\infty} w(i)z^i$ convergent for $|z| < \rho$, then, if $|a_i b_j| < \rho$ for all i, j , we have:

$$\sum_{k_1 > k_2 > \dots > k_m \geq 0} \det[a_i^{k_j}] \det[b_i^{k_j}] \prod_{i=1}^m w(k_i) = \det[W(a_i b_j)] \quad (19)$$

where the determinants are all taken with respect to the indices i and j .

These few definitions and formulae will be enough to allow us to manipulate integrals over the unitary group in extremely powerful ways.

IV. CALCULATION OF THE JOINT DISTRIBUTION OF EIGENVALUES OF $\mathbf{G}\mathbf{G}^\dagger$ USING THE CHARACTER EXPANSION

As we saw in the end of Section II-B, one can calculate $g(z)$ by first calculating the joint probability density $P(\{\lambda_i\})$ of the eigenvalues λ_i . In this section we will introduce a new method to calculate the joint probability density for Gaussian channel matrices \mathbf{G} with non-trivial correlations or non-zero mean. As we shall see, this method allows us to obtain $P(\{\lambda_i\})$ with minimal effort, using the results introduced in Section III. Given its power and simplicity, we will expose the basic steps of the approach in the main part of the paper, even though the results presented in this section have been derived elsewhere, albeit using more involved and less applicable methods.

To express the probability density in terms of the eigenvalues λ_i of $\mathbf{G}\mathbf{G}^\dagger$, it is convenient to change integration variables, from the elements of \mathbf{G} to its singular values,

$$\mu_i = \sqrt{\lambda_i} = \text{SVD}(\mathbf{G}) \quad (20)$$

for $i = 1, \dots, M$, using the standard transformation

$$\mathbf{G} = \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \quad (21)$$

where \mathbf{U} , \mathbf{V} are unitary $n_t \times n_t$ and $n_r \times n_r$ matrices, respectively and $diag(\boldsymbol{\mu})$ is a diagonal $n_t \times n_r$ matrix with diagonal elements μ_i , for $i = 1, \dots, N$. Including the Jacobian of the coordinate transformation we get the following relation [2], [32]–[34]

$$\mathcal{D}\mathbf{G} = \mathcal{C}_{M,N} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} d\boldsymbol{\lambda} D\mathbf{U} D\mathbf{V} \quad (22)$$

where $\boldsymbol{\lambda}$ is an N -dimensional vector with elements λ_i and $\Delta(\boldsymbol{\lambda})$ is the corresponding Vandermonde determinant. $d\boldsymbol{\lambda}$ is the product of the λ_i differentials $d\boldsymbol{\lambda} = d\lambda_1 \dots d\lambda_N$, while $D\mathbf{U}$ and $D\mathbf{V}$ are the standard Haar integration measures for the corresponding unitary matrices, see for example [28]. The normalization constant $\mathcal{C}_{M,N}$ is given by the Laguerre-Selberg integral (see p. 354 in [32])

$$\mathcal{C}_{M,N}^{-1} = \prod_{j=1}^N [\Gamma(j+1)\Gamma(M-N+j)] \quad (23)$$

Thus, we can express $P(\{\lambda_i\})$ as

$$P(\{\lambda_i\}) = \mathcal{C}_{M,N} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \int D\mathbf{U} \int D\mathbf{V} p(\mathbf{G} = \mathbf{U}diag(\boldsymbol{\mu})\mathbf{V}^\dagger) \quad (24)$$

where the integrals are over all unitary matrices \mathbf{U} , \mathbf{V} and $p(\mathbf{G})$ is the probability distribution of the matrix \mathbf{G} .

A. Case of i.i.d. channel with zero mean

When \mathbf{G} is independently and identically distributed (i.i.d.) with zero mean, the probability distribution is given by

$$p(\mathbf{G}) = e^{-\text{Tr}\{\mathbf{G}^\dagger \mathbf{G}\}} \quad (25)$$

Substituting $\mathbf{G} = \mathbf{U}diag(\boldsymbol{\mu})\mathbf{V}^\dagger$ we obtain

$$p(\mathbf{G}) = \exp \left[-\text{Tr} \left\{ diag(\boldsymbol{\mu})^\dagger diag(\boldsymbol{\mu}) \right\} \right] = \exp \left[-\text{Tr} \{ diag(\boldsymbol{\lambda}) \} \right] \quad (26)$$

which is independent of both \mathbf{U} and \mathbf{V} . $diag(\boldsymbol{\lambda})$ is an $N \times N$ diagonal matrix with diagonal elements $\lambda_i = \mu_i^2$, which are the eigenvalues of $\mathbf{G}^\dagger \mathbf{G}$. Thus, the integrals over the unitary group in (24) are trivial and we immediately obtain the well-known result [32]

$$P(\{\lambda_i\}) = \mathcal{C}_{M,N} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \left[\lambda_i^{M-N} e^{-\lambda_i} \right] \quad (27)$$

It is clear that this particular case is rather unique in that the unitary integrals in (24) are trivial. More generally, the argument of these integrals will contain \mathbf{U} and \mathbf{V} explicitly, making the joint distribution harder to obtain, as we shall see below.

B. Case of semi-correlated \mathbf{G} with zero mean

The simplest case of channel with a unitary matrix explicitly in $p(\mathbf{G})$ is that of a semi-correlated channel, i.e. one with non-trivial correlations on only one side of \mathbf{G} , with the probability distribution taking the form

$$p(\mathbf{G}) = \mathcal{N}_{\mathbf{T}} \exp \left[-\text{Tr} \left\{ \mathbf{G} \mathbf{T}^{-1} \mathbf{G}^\dagger \right\} \right] = \mathcal{N}_{\mathbf{T}} \exp \left[-\text{Tr} \left\{ \text{diag}(\tilde{\boldsymbol{\lambda}}) \mathbf{U}^\dagger \mathbf{T}^{-1} \mathbf{U} \right\} \right] \quad (28)$$

where the normalization constant is $\mathcal{N}_{\mathbf{T}}^{-1} = \det \mathbf{T}^{n_r}$ and $\text{diag}(\tilde{\boldsymbol{\lambda}})$ is an n_t -dimensional diagonal matrix with the first N diagonal elements $\tilde{\lambda}_i = \lambda_i$, for $i = 1, \dots, N$ and the remaining $M - N$ ones (if $n_t = M > N = n_r$) taking for now arbitrary values. In the end of the calculation we take the limit $\tilde{\lambda}_{N+1}, \dots, \tilde{\lambda}_M \rightarrow 0$. As a result, only N non-zero eigenvalues of $\mathbf{G} \mathbf{G}^\dagger$ will remain.

Inserting (28) into (24), $P(\{\lambda_i\})$ can be expressed as

$$P(\{\lambda_i\}) = \mathcal{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \int D\mathbf{U} \exp \left[-\text{Tr} \left\{ \text{diag}(\tilde{\boldsymbol{\lambda}}) \mathbf{U}^\dagger \mathbf{T}^{-1} \mathbf{U} \right\} \right] \quad (29)$$

Step 1: Integration over \mathbf{U} . We next need to integrate out the unitary matrix \mathbf{U} . Such integrals are well known [19], [21], and we could easily skip directly to the result (37). However, it is useful to go through this particular derivation, so that the reader will become familiar with the technique. Applying Lemma 2 to the exponential of (29) we get

$$\begin{aligned} P(\{\lambda_i\}) &= \mathcal{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \int D\mathbf{U} \sum_{\mathbf{m}} \alpha_{\mathbf{m}}(-1) \chi_{\mathbf{m}} \left(\text{diag}(\tilde{\boldsymbol{\lambda}}) \mathbf{U}^\dagger \mathbf{T}^{-1} \mathbf{U} \right) \\ &= \mathcal{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \int D\mathbf{U} \sum_{\mathbf{m}} \alpha_{\mathbf{m}}(-1) \text{Tr} \left\{ \text{diag}(\tilde{\boldsymbol{\lambda}})^{(\mathbf{m})} \left(\mathbf{U}^{(\mathbf{m})} \right)^\dagger \left(\mathbf{T}^{(\mathbf{m})} \right)^{-1} \mathbf{U}^{(\mathbf{m})} \right\} \end{aligned} \quad (30)$$

where $\alpha_{\mathbf{m}}(x)$ is defined in (17) and the sum is over all integer vectors $\mathbf{m} = [m_1, \dots, m_{n_t}]$ with $m_1 \geq m_2 \geq \dots \geq m_{n_t} \geq 0$, corresponding to the representations of $U(n_t)$ (see (16)). We remind the reader that we use the notation $\mathbf{M}^{(\mathbf{m})}$ to be the group element \mathbf{M} in the representation \mathbf{m} . The second equation here results from the definition of the character $\chi_{\mathbf{m}}$ as the trace of its argument. As mentioned above the integrand now is *quadratic* in the unitary matrix elements and thus we may integrate over the unitary matrix \mathbf{U} using the standard orthogonality relation between unitary matrix elements, given in Eq. (18).

As a result we get

$$\begin{aligned} P(\{\lambda_i\}) &= \mathcal{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \sum_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}(-1)}{d_{\mathbf{m}}} \text{Tr} \left\{ \left(\mathbf{T}^{(\mathbf{m})} \right)^{-1} \right\} \text{Tr} \left\{ \text{diag}(\tilde{\boldsymbol{\lambda}})^{(\mathbf{m})} \right\} \\ &= \mathcal{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \sum_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}(-1)}{d_{\mathbf{m}}} \chi_{\mathbf{m}} \left(\mathbf{T}^{-1} \right) \chi_{\mathbf{m}} \left(\text{diag}(\tilde{\boldsymbol{\lambda}}) \right) \end{aligned} \quad (31)$$

We may now use Weyl's character formula for the unitary groups (15) to represent the characters in terms of the eigenvalues of the corresponding matrices. We also use (17) to represent $\alpha_{\mathbf{m}}/d_{\mathbf{m}}$. As a result of these substitutions we get

$$P(\{\lambda_i\}) = \tilde{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} \sum_{\mathbf{m}} \left[\prod_{i=1}^{n_t} \frac{(-1)}{(m_i - i + n_t)!} \right] \frac{\det(t_i^{m_j - j + n_t}) \det(\tilde{\lambda}_i^{m_j - j + n_t})}{\Delta(\mathbf{t}) \Delta(\tilde{\boldsymbol{\lambda}})} \quad (32)$$

where the vector \mathbf{t} was introduced with

$$\mathbf{t} = \text{diag} \left(\text{eigs} \left[\mathbf{T}^{-1} \right] \right) \quad (33)$$

where $\text{eigs}[\mathbf{T}^{-1}]$ is the vector of eigenvalues of \mathbf{T}^{-1} . In addition,

$$\tilde{C}_{M,N} = \mathcal{C}_{M,N} \prod_{i=1}^{n_t} (i-1)! \quad (34)$$

From the form of (32) it is clear that the change of summation variables from m_i to k_i with

$$k_i = m_i - i + n_t \quad (35)$$

will simplify the notation. Thus the summation over the new variables is over all integer vectors $\mathbf{k} = [k_1, \dots, k_{n_t}]$, with $k_1 > k_2 > \dots > k_{n_t} \geq 0$. The resulting form of the joint probability distribution $P(\{\lambda_i\})$ is

$$P(\{\lambda_i\}) = \tilde{C}_{M,N} \mathcal{N}_{\mathbf{T}} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} (-1)^{n_t(n_t-1)/2} \sum_{\mathbf{k}} \left[\prod_{i=1}^{n_t} \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(\tilde{\lambda}_i^{k_j})}{\Delta(\mathbf{t}) \Delta(\tilde{\boldsymbol{\lambda}})} \quad (36)$$

Step 2: Resummation of series. The second step in calculating $P(\{\lambda_i\})$ is to re-sum the series in (36). Despite the apparent complexity of this multiple sum, it is in precisely of the form of the Cauchy-Binet formula, described in Lemma 3. Applying this formula with $w(k) = (-1)^k/k!$ and therefore $W(x) = \exp(-x)$ to (36) we obtain the joint eigenvalue density of eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ for the case of semicorrelated zero-mean channels:

$$P(\{\lambda_i\}) = \tilde{C}_{M,N} \prod_{i=1}^{n_t} t_i^{n_r} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^N \lambda_i^{M-N} (-1)^{n_t(n_t-1)/2} \frac{\det[e^{-t_i \tilde{\lambda}_j}]}{\Delta(\tilde{\boldsymbol{\lambda}}) \Delta(\mathbf{t})} \quad (37)$$

Step 3: Allowing for $M > N$. The final step in the calculation is to reassign $\tilde{\lambda}_j = \lambda_j$ for $j = 1, \dots, N$ and $\tilde{\lambda}_j = 0$ for $j = N+1, \dots, M$.

1) *Case $n_t = N \leq M = n_r$:* In this case (37) is essentially the final result with $\tilde{\lambda}_j = \lambda_j$, for $j = 1, \dots, N$. After some cancellations we have

$$P(\{\lambda_i\}) = \frac{1}{n_t!} \prod_{j=1}^{n_t} \frac{t_j^{n_r} \lambda_j^{n_r - n_t}}{(n_r - n_t + j - 1)!} \Delta(\boldsymbol{\lambda}) (-1)^{n_t(n_t-1)/2} \frac{\det[e^{-t_i \lambda_j}]}{\Delta(\mathbf{t})} \quad (38)$$

2) *Case* $n_t = M > N = n_r$: To take the limit $\tilde{\lambda}_{N+1}, \dots, \tilde{\lambda}_M \rightarrow 0$ we use Lemma 6, which is an application of the l'Hospital rule and appears in Appendix III. Applying this lemma to (37) with $f_i(x) = e^{-t_i x}$ we get the joint probability density $P(\{\lambda_i\})$ for $n_t > n_r$

$$P(\{\lambda_i\}) = \prod_{j=1}^{n_r} \frac{1}{j!} \prod_{i=1}^{n_t} t_i^{n_r} \Delta(\boldsymbol{\lambda}) (-1)^{n_r(n_r-1)/2} \frac{\det \left[1; t_i; \dots; t_i^{n_t-n_r-1}; e^{-t_i \lambda_1}; \dots; e^{-t_i \lambda_{n_r}} \right]}{\Delta(\mathbf{t})} \quad (39)$$

Clearly, if we interchange the correlated side from transmitter to receiver, one only needs to interchange n_r with n_t in the above equations, where again t_i are the eigenvalues of \mathbf{T}^{-1} , see (33).

Summarizing the results of this section, we have calculated the joint eigenvalue distribution for $\mathbf{G}\mathbf{G}^\dagger$ with \mathbf{G} being semicorrelated, using a single unified approach, rather than two independent methods used before, for the two limits $n_t > n_r$ [35] and $n_r > n_t$ [17].³ In addition, our method, based on the character expansion (Lemma 2), the Cauchy-Binet formula (Lemma 3), and the generalized l'Hospital rule (Lemma 6) is simple enough to be applied to other types of problems, as we shall see below.

To obtain the moment generating function $g(z)$ one needs to integrate over the λ 's, see (11). From this point one can employ the methods by [17], [18] to calculate the moment generating function. For completeness, we include the calculation of $g(z)$ in Appendix IV for both cases above.

C. Case of uncorrelated \mathbf{G} with non-zero mean \mathbf{G}_0

We will now analyze a simple case, where both unitary integrals are present in (24), namely the case of uncorrelated \mathbf{G} with non-zero mean \mathbf{G}_0 . In this case the probability density $p(\mathbf{G})$ takes the form

$$p(\mathbf{G}) = \exp \left[-\text{Tr} \left\{ (\mathbf{G} - \mathbf{G}_0)^\dagger (\mathbf{G} - \mathbf{G}_0) \right\} \right] \quad (40)$$

$$= e^{-\sum_{i=1}^N \lambda_i - \sum_{i=1}^{N_0} \gamma_i} \exp \left[\text{Tr} \left\{ \mathbf{G}_0^\dagger \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \right\} + \text{Tr} \left\{ \mathbf{V} \text{diag}(\boldsymbol{\mu})^\dagger \mathbf{U}^\dagger \mathbf{G}_0 \right\} \right] \quad (41)$$

where γ_i for $i = 1, \dots, N_0$ are the non-zero eigenvalues of $\mathbf{G}_0^\dagger \mathbf{G}_0$ and where we have used the singular value decomposition of \mathbf{G} , (20).

Step 1: Integration over \mathbf{U} , \mathbf{V} . It will be convenient below to have the integrations of both \mathbf{U} and \mathbf{V} over the same unitary group, therefore we will extend both \mathbf{G}_0 and \mathbf{G} in the second line in (41) to $M \times M$ matrices, by adding an appropriate number of columns or rows, which will eventually be set to zero. Essentially what we are doing is allowing initially for M non-zero eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ for the

³We note that had [17] applied Lemma 6 to their result, they would also had been able to get both $n_t > n_r$ and $n_t < n_r$ results.

terms in the exponent and later, if appropriate ($M \neq N$), letting the extra ones go to zero. Inserting (40) into (24) we get

$$P(\{\lambda_i\}) = C_{M,N} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^{N_0} e^{-\gamma_i} \prod_{i=1}^N [\lambda_i^{M-N} e^{-\lambda_i}] \quad (42)$$

$$\cdot \int DU \int DV \exp \left[\text{Tr} \left\{ \mathbf{G}_0^\dagger \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \right\} + \text{Tr} \left\{ \mathbf{V} \text{diag}(\boldsymbol{\mu})^\dagger \mathbf{U}^\dagger \mathbf{G}_0 \right\} \right]$$

We may now apply the character expansion Lemma 2 to each of the exponentials of traces above. As a result, the second line in the equation above becomes

$$\int DU \int DV \sum_{\mathbf{m}} \sum_{\mathbf{m}'} \alpha_{\mathbf{m}}(1) \alpha_{\mathbf{m}'}(1) \chi_{\mathbf{m}} \left(\mathbf{G}_0^\dagger \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \right) \chi_{\mathbf{m}'} \left(\mathbf{V} \text{diag}(\boldsymbol{\mu})^\dagger \mathbf{U}^\dagger \mathbf{G}_0 \right) \quad (43)$$

Since both \mathbf{V} and \mathbf{U} appear as quadratic forms in the double sum, we may apply (18) to integrate both unitary integrals. (43) thus becomes

$$\sum_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}(1)^2}{d_{\mathbf{m}}^2} \chi_{\mathbf{m}} \left(\mathbf{G}_0^\dagger \mathbf{G}_0 \right) \chi_{\mathbf{m}} \left(\text{diag}(\tilde{\boldsymbol{\lambda}}) \right) = \sum_{\mathbf{k}} \left[\frac{\prod_{j=1}^M (j-1)!}{\prod_{j=1}^M k_j!} \right]^2 \left[\frac{\det(\tilde{\boldsymbol{\gamma}}^{k_j}) \det(\tilde{\boldsymbol{\lambda}}^{k_j})}{\Delta(\tilde{\boldsymbol{\gamma}}) \Delta(\tilde{\boldsymbol{\lambda}})} \right] \quad (44)$$

where in the right hand side of the above equation we applied Weyl's character formula (15), substituted for $\alpha_{\mathbf{m}}$ using (17) and redefined the summation in terms of \mathbf{k} (35). Once again we introduced the M -dimensional vector $\tilde{\boldsymbol{\lambda}}$ with elements $\tilde{\lambda}_i = \lambda_i$ for $i = 1, \dots, N$ and (for $M > N$) $\tilde{\lambda}_{N+1}, \dots, \tilde{\lambda}_M$ arbitrary at this point. In addition, we introduced another M -dimensional vector, $\tilde{\boldsymbol{\gamma}}$, such that $\tilde{\gamma}_i = \gamma_i$ for $i = 1, \dots, N_0$ and, (assuming $N_0 < M$) for $i = N_0 + 1, \dots, M$ it is currently arbitrary, but will eventually be set to zero.

Step 2: Resummation of series. We next apply the Cauchy-Binet formula, Lemma 3 to resum the above sums, using $w(k) = 1/k!^2$ and the Taylor expansion of the modified Bessel function [36]

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left[\frac{x}{2} \right]^{2k} \quad (45)$$

(44) thus becomes

$$\prod_{j=1}^M (j-1)!^2 \frac{\det \left(I_0(2\sqrt{\tilde{\gamma}_i \tilde{\lambda}_j}) \right)}{\Delta(\tilde{\boldsymbol{\gamma}}) \Delta(\tilde{\boldsymbol{\lambda}})} \quad (46)$$

Inserting this into (42) we obtain

$$P(\{\lambda_i\}) = \frac{\prod_{j=1}^{M-N} [(j-1)!(N+j-1)!]}{N!} \Delta(\boldsymbol{\lambda})^2 \prod_{i=1}^{N_0} e^{-\gamma_i} \prod_{i=1}^N [\lambda_i^{M-N} e^{-\lambda_i}] \frac{\det \left(I_0(2\sqrt{\tilde{\gamma}_i \tilde{\lambda}_j}) \right)}{\Delta(\tilde{\boldsymbol{\gamma}}) \Delta(\tilde{\boldsymbol{\lambda}})} \quad (47)$$

Step 3: Allowing for $M > N$ and $M > N_0$. As in the previous section, we may now let $\tilde{\gamma}_i = \gamma_i$ for $i = 1, \dots, N_0$ and $\tilde{\lambda}_i = \lambda_i$ for $i = 1, \dots, N$ and if $M > N$ and/or $M > N_0$ we may apply Lemma 6

with the parameter $x_0 = 0$ to let the superfluous $\tilde{\gamma}_i$ for $i = N_0 + 1, \dots, M$ and $\tilde{\lambda}_i$ for $i = N + 1, \dots, M$ go to zero. The final expression for $P(\{\lambda_i\})$ reads

$$P(\{\lambda_i\}) = \frac{(-1)^{(N_0+N)(M-1)}}{N!} \frac{\Delta(\boldsymbol{\lambda}) \det \mathbf{L}}{\Delta(\boldsymbol{\gamma})} \prod_{i=1}^N e^{-\lambda_i} \prod_{j=1}^{N_0} e^{-\gamma_j} \frac{\prod_{i=1}^{M-N} \left[\frac{(N+i-1)!}{(i-1)!} \right]}{\prod_{j=1}^{M-N_0} [(j-1)!]^2 \prod_{j=1}^{N_0} \gamma_j^{M-N_0}} \quad (48)$$

where it is assumed that in the last ratio the numerator is unity when $M = N$, while the denominator is unity when $M = N_0$. \mathbf{L} is a $M \times M$ matrix with elements

$$L_{ij} = \begin{cases} I_0(2\sqrt{\gamma_i \lambda_j}) & i \leq N_0, j \leq N \\ \lambda_j^{i-N_0-1} & i > N_0, j \leq N \\ \gamma_i^{j-N-1} & i \leq N_0, j > N \\ ((i-N_0-1)!)^2 \delta_{i-N_0, j-N} & i > N_0, j > N \end{cases} \quad (49)$$

This obtains the result by [15] using simple Bessel functions rather than more complicated hypergeometric functions, with only a few simple steps involved. Using this result, we calculate $g(z)$ in Appendix IV.

D. Case of fully correlated \mathbf{G} with zero mean

In the previous sections we recovered already known results, albeit in a very direct and simple way. We now turn to a problem that has not been previously solved. In this section we will analyze the joint probability distribution of eigenvalues of $\mathbf{G}^\dagger \mathbf{G}$ when \mathbf{G} has zero mean but is fully correlated, i.e. its probability density is

$$p(\mathbf{G}) = \mathcal{N}_{\mathbf{T}, \mathbf{R}} \exp \left[-\text{Tr} \left\{ \mathbf{G}^\dagger \mathbf{T}^{-1} \mathbf{G} \mathbf{R}^{-1} \right\} \right] \quad (50)$$

with

$$\mathcal{N}_{\mathbf{T}, \mathbf{R}}^{-1} = \det \mathbf{T}^{n_r} \det \mathbf{R}^{n_t} \quad (51)$$

Applying (21) to (50) and then inserting it into (24) we get

$$\begin{aligned} P(\{\lambda_i\}) &= \mathcal{N}_{\mathbf{T}, \mathbf{R}} \mathcal{C}_{M,N} \Delta(\boldsymbol{\lambda})^2 \int D\mathbf{U} \int D\mathbf{V} \exp \left[-\text{Tr} \left\{ \mathbf{T}^{-1} \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \mathbf{R}^{-1} \mathbf{V} \text{diag}(\boldsymbol{\mu})^\dagger \mathbf{U}^\dagger \right\} \right] \\ &= \mathcal{N}_{\mathbf{T}, \mathbf{R}} \Delta(\boldsymbol{\lambda}) \mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda}) \end{aligned} \quad (52)$$

where the second equation defines the quantity $\mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda})$. At this point it is convenient to assume that $M = N$, relaxing the constraint later.

Step 1: Integration over \mathbf{U} , \mathbf{V} . We may now use Lemma 2 to expand the exponential so that the integration over \mathbf{U} , \mathbf{V} can be performed. As a result we get

$$\mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda}) = \mathcal{C}_{M,N} \Delta(\boldsymbol{\lambda}) \sum_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}(-1)}{d_{\mathbf{m}}^2} \chi_{\mathbf{m}}(\mathbf{T}^{-1}) \chi_{\mathbf{m}}(\boldsymbol{\lambda}) \chi_{\mathbf{m}}(\mathbf{R}^{-1}) \quad (53)$$

where $\alpha_{\mathbf{m}}$ is the expansion coefficient, see (17), and $d_{\mathbf{m}}$ is the dimension of the \mathbf{m} representation, see (12). Comparing the above equation to (31), (44) we see a crucial difference here, namely that $\alpha_{\mathbf{m}}$, $d_{\mathbf{m}}$ do not appear with the same powers. This is important because, using (17) we can no longer cancel $d_{\mathbf{m}}$, which as seen (12) is a rather complicated function. Applying Weyl's character formula (15) and redefining the summations from \mathbf{m} to \mathbf{k} , where $k_i = m_i - i + M$ similar to (35) we get

$$\mathcal{I}(\mathbf{T}, \mathbf{R}, \lambda) = \frac{1}{M!} \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(\lambda_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{k})\Delta(\mathbf{t})\Delta(\mathbf{r})} \quad (54)$$

where we also used Lemma 1 to express $d_{\mathbf{k}}$ in terms of $\Delta(\mathbf{k})$ and we introduced the vectors \mathbf{t} , \mathbf{r} , with elements the inverse eigenvalues of \mathbf{T} , \mathbf{R} , respectively, i.e.

$$\begin{aligned} \mathbf{t} &= \text{diag} \left(\text{eigs} \left[\mathbf{T}^{-1} \right] \right) \\ \mathbf{r} &= \text{diag} \left(\text{eigs} \left[\mathbf{R}^{-1} \right] \right) \end{aligned} \quad (55)$$

Unfortunately, we may not make any further progress in resumming the above expression, due to two impediments. First, to apply Cauchy-Binet's formula, Lemma 3, we need two determinants of the form $\det(x_i^{k_j})$ in the numerator. Instead we have three. In addition, the extra $\Delta(\mathbf{k})$ in the denominator does not allow us to separate the different k_i sums. However, we can still get the moment generating function for the mutual information $g(z)$, by first integrating over λ_i . This will be done in the next section.

V. MOMENT-GENERATING FUNCTION FOR \mathbf{G} FULLY CORRELATED

Even if in Section IV-D we were not able to find an final expression for the joint probability of eigenvalues of $\mathbf{G}^\dagger \mathbf{G}$ for the case of fully correlated \mathbf{G} , in this section we will generate a closed-form expression for the moment generating function of the mutual information, $g(z)$, using (54).

A. $g(z)$ for the case $M = N$

From (11) $g(z)$ can be written (for $M = N$) as

$$g(z) = \int d\lambda \prod_{i=1}^M (1 + \lambda_i)^z P(\{\lambda_i\}) \quad (56)$$

where the integral is over positive λ_i 's, and P is the joint probability distribution function of the eigenvalues. Inserting (54) into (52) $g(z)$ can be written as

$$g(z) = \frac{\mathcal{N}_{\mathbf{T}, \mathbf{R}}}{M! \Delta(\mathbf{t}) \Delta(\mathbf{r})} \int d\lambda \prod_{i=1}^M (1 + \lambda_i)^z \Delta(\lambda) \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(r_i^{k_j}) \det(\lambda_i^{k_j})}{\Delta(\mathbf{k})} \quad (57)$$

Step 2: Integration over $\{\lambda_i\}$. As mentioned above, we cannot resum the series in the form (57). Instead, we will first integrate over λ_i . To proceed, it is useful to transform the Vandermonde determinant $\Delta(\boldsymbol{\lambda})$ into another one with arguments $\lambda_i/(1 + \lambda_i)$

$$\begin{aligned}
\Delta(\boldsymbol{\lambda}) &= \prod_{i>j} [\lambda_i - \lambda_j] = \prod_{i>j} [(1 + \lambda_i) - (1 + \lambda_j)] \\
&= \left[\prod_{m=1}^M (1 + \lambda_m)^{M-1} \right] \prod_{i>j} \left(\frac{1}{1 + \lambda_j} - \frac{1}{1 + \lambda_i} \right) \\
&= \left[\prod_{m=1}^M (1 + \lambda_m)^{M-1} \right] \prod_{i>j} \left(\frac{\lambda_i}{1 + \lambda_i} - \frac{\lambda_j}{1 + \lambda_j} \right) \\
&= \left[\prod_{m=1}^M (1 + \lambda_m)^{M-1} \right] \det \left[\left(\frac{\lambda_i}{1 + \lambda_i} \right)^{j-1} \right]
\end{aligned} \tag{58}$$

The usefulness of this form of the determinant will become apparent in a moment. We first expand the above determinant using $\text{sgn}(\mathbf{b})$ as in (2). Thus $g(z)$ becomes

$$g(z) = \mathcal{N}_{\mathbf{TR}} \sum_{\mathbf{b}} \text{sgn}(\mathbf{b}) \int d\boldsymbol{\lambda} \left\{ \prod_{i=1}^M (1 + \lambda_i)^{z+M-b_i} \right\} \left[\prod_{j=1}^M \lambda_j^{b_j-1} \mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda}) \right] \tag{59}$$

We now wish to integrate the above expression by parts with respect to the λ 's, integrating the curly brackets and differentiating the square brackets. Usually, when integrating by parts we generate various boundary terms evaluated at $\lambda = 0$ and $\lambda = \infty$. For example, if we integrate by parts p times over a particular λ , we will get a sum over p terms, in which the n th term (with $n = 1, \dots, p$) has the square brackets differentiated $n - 1$ times with respect to this λ and is evaluated at $\lambda = 0$ and $\lambda = \infty$. However, here we will point out that these boundary terms are zero in the cases we are concerned with.

We note that according to Lemma 5 in Appendix II the quantity $\mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda})$ is exponentially convergent when any λ_i approaches infinity. Therefore, the square bracketed quantity above, as well as any of its derivatives with respect to the λ 's is zero when evaluated at any $\lambda_i = \infty$. As a result, we may discard all generated boundary terms at $\lambda_i = \infty$. In addition, we observe from its definition in (54) that \mathcal{I} is finite when any of the λ 's approach zero. Thus if we integrate each λ_i up to $b_i - 1$ times by parts in (59), all boundary terms evaluated at $\lambda_i = 0$ for $i = 1, \dots, M$, will vanish, since they involve derivatives of the square-bracketed quantity above, not exceeding $b_i - 1$ order. Indeed, we will choose to integrate by parts precisely $b_i - 1$ times in order to make further progress.

To proceed with this multiple integration by parts, we first note that the $(b_i - 1)$ th indefinite integral of the curly brackets in (59) with respect to all λ_i is

$$\prod_{i=1}^M \frac{(1 + \lambda_i)^{z+M-1}}{\prod_{p=1}^{b_i-1} (z + M - p)} \tag{60}$$

with the product in the denominator assumed to be unity for $b_i = 1$. To obtain the $(b_i - 1)$ th derivative of the square brackets in (59) we need to rewrite \mathcal{I} as an expansion in the form of (54). In particular, using $\text{sgn}(\mathbf{c})$ (see (2)) to expand $\det(\lambda_i^{k_j})$ in (54), we can rewrite the square brackets of (59) as

$$[\cdot] = \frac{1}{M!} \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{k})\Delta(\mathbf{t})\Delta(\mathbf{r})} \sum_{\mathbf{c}} \text{sgn}(\mathbf{c}) \prod_{j=1}^M \lambda_j^{k_{c_j} + b_j - 1} \quad (61)$$

where the sum over \mathbf{c} is over all permutations of $(1, 2, \dots, M)$. By differentiating all λ_i in the above expression $b_i - 1$ times we get

$$\frac{1}{M!} \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{k})\Delta(\mathbf{t})\Delta(\mathbf{r})} \sum_{\mathbf{c}} \text{sgn}(\mathbf{c}) \prod_{j=1}^M \left[\lambda_j^{k_{c_j}} \prod_{p=1}^{b_j-1} (k_{c_j} + p) \right] \quad (62)$$

Combining all above results, $g(z)$ of (59) can be expressed as

$$g(z) = \frac{\mathcal{N}_{\mathbf{TR}}}{M!} \int d\lambda \prod_{i=1}^M (1 + \lambda_i)^{z+M-1} \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{k})\Delta(\mathbf{t})\Delta(\mathbf{r})} \quad (63)$$

$$\times \sum_{\mathbf{b}, \mathbf{c}} \text{sgn}(\mathbf{b})\text{sgn}(\mathbf{c}) \prod_{j=1}^M \left[(-1)^{b_j-1} \lambda_j^{k_{c_j}} \prod_{p=1}^{b_j-1} \frac{k_{c_j} + p}{z + M - p} \right]$$

where the additional $(-1)^{b_j-1}$ comes from the integration by parts.

We will now integrate over the λ_i 's *before* summing over the \mathbf{k} 's. Looking at the above equation this seems problematic since each of the terms in the sum over \mathbf{k} above is unbounded at large λ . However, from Lemma 5 in Appendix II we know that \mathcal{I} and thus the integrand above is exponentially bounded at large λ and hence integrable. To circumvent the discrepancy we introduce a cutoff function $f(\lambda)$ for every λ , which is unity at $\lambda \rightarrow 0$ and tends to zero faster than a power law as $\lambda \rightarrow \infty$. For example, we can take $f(x) = e^{-\delta x}$. In the end of the calculation we will set $\delta = 0$ and thus make $f(x)$ unity for all x . Cutting off the integrals makes all terms finite and thus we can freely interchange the order of the summation and integration. Thus we have

$$g(z) = \frac{\mathcal{N}_{\mathbf{TR}}}{M!} \sum_{\mathbf{k}} \left[\prod_{i=1}^M \frac{(-1)^{k_i}}{k_i!} \right] \frac{\det(t_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{k})\Delta(\mathbf{t})\Delta(\mathbf{r})} \quad (64)$$

$$\times \sum_{\mathbf{b}, \mathbf{c}} \text{sgn}(\mathbf{b})\text{sgn}(\mathbf{c}) \prod_{i=1}^M \left\{ (-1)^{b_i-1} \prod_{p=1}^{b_i-1} \frac{k_{c_i} + p}{z + M - p} \int_0^\infty d\lambda_i \lambda_i^{k_{c_i}} (1 + \lambda_i)^{z+M-1} f(\lambda_i) \right\}$$

Next it is important to appreciate the following property of the second line of the above equation. Once the integrals over λ_i have been performed, the quantities in the curly bracket depend on the index i only through the indices b_i and c_i . Therefore, the second line of the above equation is in the form of (2) and

thus it can be written as $M!$ times a determinant of the matrix \mathbf{W} with elements

$$\begin{aligned} W_{ij} &= \prod_{p=1}^{i-1} \frac{k_j + p}{z + M - p} (-1)^{i-1} \int_0^\infty d\lambda \lambda^{k_j} (1 + \lambda)^{z+M-1} f(\lambda) \\ &= \prod_{p=1}^{i-1} \frac{k_j + p}{z + M - p} (-1)^{i-1} W_{1j} \end{aligned} \quad (65)$$

where the second equation results from the fact that the only i -dependence of W_{ij} is outside the integral.

Thus W_{1j} is simply the integral above and can be factored outside the determinant, resulting in

$$\det \mathbf{W} = \prod_{j=1}^M W_{1j} \det \left[\prod_{p=1}^{i-1} \frac{k_j + p}{z + M - p} (-1)^{i-1} \right] \quad (66)$$

$$= (-1)^{M(M-1)/2} \prod_{j=1}^M \frac{W_{1j}}{(z + j - 1)^{j-1}} \det \left(\frac{(k_j + i - 1)!}{k_j!} \right) \quad (67)$$

In (67), the power of (-1) comes from bringing $(-1)^{i-1}$ outside the determinant, while the product of $(z + j - 1)^{j-1}$ originates from the denominator in the determinant.

Using the same procedure as in Lemma 1 in Appendix I, we can show that the determinant as in the right-hand side of (67) is simply equal to $(-1)^{M(M-1)/2} \Delta(\mathbf{k})$. We can thus rewrite the second line in (64) as

$$\prod_{i=1}^M \frac{\int_0^\infty d\lambda \lambda^{k_i} (1 + \lambda)^{z+M-1} f(\lambda)}{(z + i - 1)^{i-1}} \Delta(\mathbf{k}) \quad (68)$$

Inserting this into (64) we see immediately that $\Delta(\mathbf{k})$ cancels leaving

$$g(z) = \mathcal{N}_{\text{TR}} \prod_{i=1}^M \frac{1}{(z + i - 1)^{i-1}} \sum_{\mathbf{k}} \frac{\det(t_i^{k_j}) \det(r_i^{k_j})}{\Delta(\mathbf{t})\Delta(\mathbf{r})} \prod_{i=1}^M \left[\int_0^\infty d\lambda \frac{(-\lambda)^{k_i}}{k_j!} (1 + \lambda)^{z+M-1} f(\lambda) \right] \quad (69)$$

Step 3: Resummation of series. We may now use the Cauchy-Binet theorem (see Lemma 3) to the above form of the sum over \mathbf{k} , which gives

$$\det \left[\int_0^\infty d\lambda e^{-r_i t_j \lambda} (1 + \lambda)^{z+M-1} f(\lambda) \right] \quad (70)$$

$$= \det \left[\int_0^\infty d\lambda e^{-r_i t_j \lambda} (1 + \lambda)^{z+M-1} \right]$$

$$= \prod_{i=1}^M \frac{1}{(t_i r_i)^M} \det F(r_i t_j; z) \quad (71)$$

In the second line above we discarded the function $f(x)$, since the integral is now convergent. I.e., we now take the limit $f(x) = \lim_{\delta \rightarrow 0} e^{-\delta x} \rightarrow 1$. Thus, $F(x, z)$ is given by [37]

$$F(x, z) = x^M \int_0^\infty d\lambda e^{-x\lambda} (1 + \lambda)^{z+M-1} = x^{-z} e^x \Gamma(z + M, x) \quad (72)$$

where $\Gamma(\alpha, x)$ is the incomplete Γ function. In summary, the moment generating function of the mutual information $g(z)$ can be written for $n_t = n_r = M = N$ as

$$g(z) = \frac{1}{\prod_{i=1}^M (z + i - 1)^{i-1} \Delta(\mathbf{t}) \Delta(\mathbf{r})} \det F(t_i r_j; z) \quad (73)$$

where the normalization $\mathcal{N}_{\mathbf{T}, \mathbf{R}}$ has been cancelled with the term in front of the determinant in (71). We note that the above expression is symmetric in \mathbf{T} and \mathbf{R} for $M = N$.

B. $g(z)$ for the case $M > N$

Step 4: Allowing for $M > N$. Next, we will obtain the expression for $M \neq N$, and specifically, without loss of generality we will assume that $n_t > n_r$, i.e. $M = n_t$, $N = n_r$. This can be readily obtained by first assuming that \mathbf{R} is M -dimensional with the first N eigenvalues the ones corresponding to the correlations at the receiver and then taking the remaining $M - N$ eigenvalues to 0. Thus the $M \neq N$ problem is obtained by letting the $M - N$ eigenvalues of one of the correlation matrices go to zero. In the particular situation, we will let r_j , for $j = N + 1, \dots, M$ go to infinity.

Even though both numerator and denominator of (73) are unbounded in this case, using Lemma 7 in Appendix III we can see that the ratio is well-defined and finite. Specifically, we see that the function $f_i(x_j)$ in the Lemma is $F(t_i r_j, z)$. Noting that the asymptotic expansion of $F(x, z)$ is (see [37], Equation 8.357)

$$F(x, z) \approx x^{M-1} \sum_{k=0}^{\infty} \frac{\Gamma(M+z)}{\Gamma(M+z-k)} x^{-k} \quad (74)$$

we find that when $r_{N+1}, \dots, r_M \rightarrow \infty$ the ratio becomes

$$\frac{\det F(t_i r_j; z)}{\Delta(\mathbf{r}_M)} \rightarrow \prod_{j=1}^{M-N-1} (M+z-j)^{M-N-j} \frac{\det \mathbf{L}}{\Delta(\mathbf{r}_N)} \quad (75)$$

where we used the notation \mathbf{r}_N , \mathbf{r}_M to explicitly specify the length of the vectors and \mathbf{L} is a M -dimensional matrix with elements

$$L_{ij} = \begin{cases} F(t_i r_j; M+z) & j \leq N \\ t_i^{j-1} & j > N \end{cases} \quad (76)$$

Summarizing all results above, we thus have the general form of the moment generating function $g(z)$ of the mutual information I for general M, N which is

$$g(z) = E_{\mathbf{G}} \left[e^{zI} \right] = \frac{\prod_{j=1}^{M-N-1} (M+z-j)^{M-N-j} \det \mathbf{L}}{\prod_{i=1}^M (z+i-1)^{i-1} \Delta(\mathbf{t}) \Delta(\mathbf{r})} \quad (77)$$

with the elements of \mathbf{L} given by (76) for $n_t \geq n_r$ and with t_i and r_i interchanged for $n_r > n_t$.

VI. ERGODIC CAPACITY FOR FULLY CORRELATED CHANNELS

In this section we will derive an expression for the ergodic capacity in the case of fully correlated channels discussed in the previous section. Without loss of generality, we will assume that $n_r = N \leq M = n_t$, as is implicitly assumed in (77). We apply (8) and differentiate $g(z)$ in (77) with respect to z :

$$g'(z) = g(z) \left(\sum_{j=1}^{M-N-1} \frac{M-N-j}{M+z-j} - \sum_{i=1}^{M-1} \frac{i}{z+i} + \frac{1}{\det \mathbf{L}} \frac{\partial \det \mathbf{L}}{\partial z} \right) \quad (78)$$

We next have to set $z = 0$, to get $E[I]$, as in (8), recalling that $g(0) = 1$, see (7). This results to

$$E[I] = g'(0) = \sum_{j=1}^{M-N-1} \frac{j}{N+j} + \sum_{j=1}^N \frac{\det \tilde{\mathbf{L}}_j}{\det \mathbf{L}} - M + 1 \quad (79)$$

where the matrix $\tilde{\mathbf{L}}_j$ is the matrix \mathbf{L} with its j -th column replaced by the column $D_{1,j}$, i.e.

$$\tilde{\mathbf{L}}_j = [L_{i,1}; L_{i,2}; \dots; L_{i,j-1}; D_{1,j}; L_{i,j+1}; \dots] \quad (80)$$

where the element $D_{i,j}$ is given by

$$\begin{aligned} D_{ij} &= \left. \frac{\partial F(t_i r_j; M+z)}{\partial z} \right|_{z=0} \\ &= \left(e^x x^M (-1)^M \left[\frac{Ei(-x)}{x} \right]^{(M-1)} \right) \Big|_{x=t_i r_j} \end{aligned} \quad (81)$$

where the superscript $(M-1)$ in the square brackets denotes the $M-1$ st derivative and $Ei(x)$ is the exponential integral [37]. Note that the second sum in (79) goes up to N , since for $j > N$ the columns in \mathbf{L} do not depend on z , as seen in (77). Similarly one can calculate higher moments of the mutual information distribution.

A. Example

We will now apply the ergodic capacity equation (79) derived above to evaluate the mutual information for the a simple case of MIMO correlated channels. Specifically, we will use the following simple model for the dependence of the correlation coefficient T_{ab} on the antenna separation and angle spread. For concreteness, we assume that the antennas form a uniform linear ideal antenna array with $d_\lambda = d_{min}/\lambda$ the nearest neighbor antenna spacing in wavelengths and we assume a Gaussian power azimuth spectrum (with 2 dimensional propagation), i.e. the average incoming power at the antenna array is $P(\theta) \propto \exp[-(\theta/\delta)^2/2]$, [38], [39] where δ is the angle spread in degrees measured from the vertical to the array. This results in a \mathbf{T} matrix with elements

$$T_{ab} = \int_{-180}^{180} \frac{d\phi}{\sqrt{2\pi\delta^2}} e^{2\pi i(a-b)d_\lambda \sin(\phi\pi/180) - \phi^2/(2\delta^2)} \quad (82)$$

with $a, b = 1 \dots n_t$ being the index of transmitting antennas. Two situations will be considered: First, where the same angle-spread appears in both transmitter and receiver, and second where only the transmitter has the above correlations, with the receiver having uncorrelated antennas. For simplicity, we assume that the transmitted signal covariance is unity, that the noise is uncorrelated and that the signal to noise ratio per transmitter antenna is unity. In this case, one can evaluate the eigenvalues of the correlation matrices and insert them in the appropriate capacity equations. Fig. 1 shows the comparison of the analytically obtained ergodic mutual information for fully correlated and semicorrelated channels with simulations. In Fig. 2 we plot the outage mutual information obtained for a number of representative cases as calculated numerically using (10) and compare them with Monte Carlo simulations. It should be stressed however, that results obtained analytically are exact, and we only include Monte-Carlo simulations as a cross-check.

VII. CONCLUSION

In conclusion, we have demonstrated a promising method recently derived in the field of representations of Lie groups, which can be used to calculate complex integrals over unitary groups. We applied this technique to recover in a simple way the joint probability distribution of eigenvalues of the matrix $\mathbf{G}\mathbf{G}^\dagger$, with \mathbf{G} being semicorrelated or having a non-zero mean, which can then be used to calculate the moment generating function of the mutual information for such Gaussian MIMO channels. In addition, we used this method to calculate the moment generating function of the mutual information for fully correlated channels. From this moment generating function we obtained the ergodic average of the mutual information and evaluated the outage probability. We believe these methods are general and applicable to a wide range of multi-antenna communications problems. As an example, in Appendix V we also analyzed their application to unitary encoded space-time transmission of MIMO systems, calculating, as a particular example, the received signal distribution when the channel matrix is correlated at the transmitter end. We expect there will be further applications of these techniques in the future.

APPENDIX I

PROOF OF LEMMA 1

Proof: We start by writing $d_{\mathbf{m}}$ as

$$d_{\mathbf{m}} = \left[\prod_{i=1}^M \frac{1}{(M-i)!} \right] \det \left[\frac{k_i!}{(k_i - M + j)!} \right] \quad (83)$$

using (14). The i, j -th element of the matrix in the determinant can be expanded to

$$k_i(k_i - 1) \dots (k_i - M + j + 1) \quad (84)$$

For example, the M th column has elements equal to unity, the $(M - 1)$ th column has elements equal to k_i , etc. We note that the product in (84) in fact vanishes when $k_i < M - j$, as noted after (12). As a result, the above expression is valid for every i, j and k_i . If we expand the above expression in powers of k_i we get

$$\sum_{p=1}^{M-j} a_{pj} k_i^{M-j-p+1} \quad (85)$$

with the coefficient $a_{1j} = 1$ for all $j = 1, \dots, M$ and all other coefficients independent of i . We now subtract $a_{M-j,j} k_i$, i.e. a multiple of column $M - 1$ from all columns $j = 1, \dots, (M - 2)$. This operation leaves the value of the determinant invariant, but reduces the upper limit in the summation in (85) to $M - j - 1$. We may now continue this process of subtracting off $a_{p,j} k_i^{M-j-p+1}$ for $p = (M - j - 1), (M - j - 2), \dots, 2$ until the only term appearing in the ij th term is k_i^{M-j} . This is exactly the form of the Vandermonde matrix, with columns j and $M - j + 1$ interchanged. Thus by reshuffling the columns to get the usual ordering, and thereby picking up a term $(-1)^{M(M-1)/2}$, we arrive to the form of (13). ■

APPENDIX II

In this appendix we will show that the quantity $\mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda})$ defined in (52) as

$$\mathcal{I} = \mathcal{C}_M \Delta(\boldsymbol{\lambda}) \int D\mathbf{U} \int D\mathbf{V} \exp \left[-\text{Tr} \left\{ \mathbf{T}^{-1} \mathbf{U} \text{diag}(\boldsymbol{\mu}) \mathbf{V}^\dagger \mathbf{R}^{-1} \mathbf{V} \text{diag}(\boldsymbol{\mu})^\dagger \mathbf{U}^\dagger \right\} \right] \quad (86)$$

is exponentially small when any of the λ 's becomes arbitrarily large. To prove this we start by showing that

Lemma 4: For any two M -dimensional positive definite matrices \mathbf{A}, \mathbf{B}

$$\text{Tr} \{ \mathbf{A} \mathbf{B} \} \geq a_{\min} \text{Tr} \{ \mathbf{B} \} \quad (87)$$

where a_{\min} is the minimum eigenvalue of \mathbf{A} . Equality holds when \mathbf{A} is proportional to the unit matrix.

Proof: Let a_1, \dots, a_M be the eigenvalues of \mathbf{A} and $\mathbf{U}_\mathbf{A}$ be the unitary matrix diagonalizing \mathbf{A} , and b_1, \dots, b_M and $\mathbf{U}_\mathbf{B}$ the corresponding eigenvalues and unitary matrix for \mathbf{B} , i.e., $\mathbf{A} = \mathbf{U}_\mathbf{A} \text{diag}(\mathbf{a}) \mathbf{U}_\mathbf{A}^\dagger$

and $\mathbf{B} = \mathbf{U}_B \text{diag}(\mathbf{b}) \mathbf{U}_B^\dagger$. Then

$$\begin{aligned}
\text{Tr}\{\mathbf{A}\mathbf{B}\} &= \text{Tr}\left\{\text{diag}(\mathbf{a})\mathbf{U}_A^\dagger\mathbf{U}_B\text{diag}(\mathbf{b}^{1/2})\left(\mathbf{U}_A^\dagger\mathbf{U}_B\text{diag}(\mathbf{b}^{1/2})\right)^\dagger\right\} \\
&= \sum_{i=1}^M a_i \sum_{j=1}^M \left| \left[\mathbf{U}_A^\dagger \mathbf{U}_B \text{diag}(\mathbf{b}^{1/2}) \right]_{ij} \right|^2 \\
&\geq a_{\min} \sum_{i=1}^M \sum_{j=1}^M \left| \left[\mathbf{U}_A^\dagger \mathbf{U}_B \text{diag}(\mathbf{b}^{1/2}) \right]_{ij} \right|^2 \\
&= a_{\min} \text{Tr}\left\{\mathbf{U}_A^\dagger \mathbf{B} \mathbf{U}_A\right\} \\
&= a_{\min} \text{Tr}\{\mathbf{B}\}
\end{aligned} \tag{88}$$

where $\text{diag}(\mathbf{b}^{1/2})$ is a diagonal matrix with diagonal elements $b_i^{1/2}$, for $i = 1, \dots, M$. The inequality holds since the term multiplying each a_i is non-negative. One gets an equality only if \mathbf{A} is proportional to the unit matrix. ■

We now apply this result to the trace in the exponent of (86)

$$\begin{aligned}
\text{Tr}\left\{\mathbf{T}^{-1}\mathbf{U}\text{diag}(\boldsymbol{\mu})\mathbf{V}^\dagger\mathbf{R}^{-1}\mathbf{V}\text{diag}(\boldsymbol{\mu})^\dagger\mathbf{U}^\dagger\right\} &\geq t_{\min}\text{Tr}\left\{\mathbf{U}\text{diag}(\boldsymbol{\mu})\mathbf{V}^\dagger\mathbf{R}^{-1}\mathbf{V}\text{diag}(\boldsymbol{\mu})^\dagger\mathbf{U}^\dagger\right\} \\
&= t_{\min}\text{Tr}\left\{\mathbf{R}^{-1}\mathbf{V}\text{diag}(\boldsymbol{\lambda})\mathbf{V}^\dagger\right\} \\
&\geq t_{\min}r_{\min}\text{Tr}\{\text{diag}(\boldsymbol{\lambda})\}
\end{aligned} \tag{89}$$

where t_{\min} and r_{\min} are the minimum eigenvalues of \mathbf{T}^{-1} , \mathbf{R}^{-1} respectively (or the inverses of the maximum eigenvalues of \mathbf{T} , \mathbf{R} , which are assumed non-zero). As a result,

$$\begin{aligned}
\mathcal{I}_{\mathbf{T},\mathbf{R},\boldsymbol{\lambda}} &\leq C_M \Delta(\boldsymbol{\lambda}) \int D\mathbf{U} \int D\mathbf{V} \exp[-t_{\min}r_{\min}\text{Tr}\{\text{diag}(\boldsymbol{\lambda})\}] \\
&= C_M \Delta(\boldsymbol{\lambda}) \exp\left[-t_{\min}r_{\min} \sum_{i=1}^M \lambda_i\right]
\end{aligned} \tag{90}$$

Since $t_{\min}r_{\min} > 0$ we finally have

Lemma 5: The quantity $\mathcal{I}(\mathbf{T}, \mathbf{R}, \boldsymbol{\lambda})$ in (86) is bounded by an exponential function of any λ_i when for any $i = 1, \dots, M$, λ_i becomes arbitrarily large.

APPENDIX III

LIMIT OF RATIOS OF FUNCTIONS

This appendix concerns limits of ratios of the form

$$R([x_1, \dots, x_M]) = \frac{\det[f_i(x_j)]}{\Delta([x_1, \dots, x_M])} \tag{91}$$

when several x_j 's become equal. In the above equation, $i, j = 1, \dots, M$, $\Delta(\mathbf{x})$ is the Vandermonde determinant and $\det[f_i(x_j)]$ is the determinant of an $M \times M$ matrix whose i, j -th element is $f_i(x_j)$.

Lemma 6: When two or more of the x_j 's in (91) become equal, then both numerator and denominator are zero. However, the ratio is generally well defined. In particular,

$$\lim_{x_1, x_2, \dots, x_p \rightarrow x_0} R([x_1, \dots, x_M]) = \frac{\det \left[f_i(x_0); f_i^{(1)}(x_0); \dots; f_i^{(p-1)}(x_0); f_i(x_{p+1}); f_i(x_{p+2}); \dots; f_i(x_M) \right]}{\Delta(x_{p+1} \dots x_M) \prod_{i=p+1}^M (x_i - x_0)^p \prod_{j=1}^{p-1} j!} \quad (92)$$

where $f^{(k)}$ represents the k^{th} derivative of the function f . (For clarity, we separate the columns by semicolons, and note that the different rows of the determinant correspond to different values of $i = 1, \dots, M$).

Proof: Proof will be by induction on p . The theorem is trivial for $p = 1$. Suppose the theorem holds up to some given p . Then we would like to take the limit as $x_{p+1} \rightarrow x_0$. Examining the matrix of the numerator, we write the elements of the $p + 1$ 'st column in a Taylor expansion

$$f_i(x_{p+1}) = \sum_{j=0}^{\infty} \frac{f_i^{(j)}(x_0)}{j!} (x_{p+1} - x_0)^j \quad (93)$$

We will now use the fact that you can subtract any multiple of one column from another column without changing the value of the determinant. Since the j^{th} column is $f_i^{(j)}(x_0)$ we can thus subtract off the first p terms of this sum ($j = 0 \dots p - 1$) without changing the determinant, leaving us with the $p + 1$ 'st column given by

$$f_i(x_{p+1}) \rightarrow f_i^{(p)}(x_0) \frac{(x_{p+1} - x_0)^p}{p!} + \text{higher powers of } (x_{p+1} - x_0) \quad (94)$$

We now examine the denominator in (92). Again, we assumed that the theorem holds for some given p . Thus, as we take $x_{p+1} \rightarrow x_0$ the denominator becomes

$$\lim_{x_{p+1} \rightarrow x_0} \Delta(x_{p+1} \dots x_M) \prod_{i=p+1}^M (x_i - x_0)^p \prod_{j=1}^{p-1} j! = (x_{p+1} - x_0)^p \left[\prod_{i=p+2}^M (x_i - x_{p+1}) \right] \Delta(x_{p+2} \dots x_M) \prod_{i=p+2}^M (x_i - x_0)^p \prod_{j=1}^{p-1} j! \quad (95)$$

Plugging in the results of (95) and (94) into (92) proves the lemma for $p + 1$. ■

A related result can be shown for the case when some elements of the vector \mathbf{x} go to infinity:

Lemma 7: If the asymptotic behavior of $f_i(x_j)$, when $x_j \rightarrow \infty$ is of the form

$$f_i(x_j) \approx x_j^{M-1} \sum_{k=0}^{\infty} \hat{f}_i^{(k)} x_j^{-k} \quad (96)$$

then we have the following limit

$$\lim_{x_1, x_2, \dots, x_p \rightarrow \infty} R(x_1, \dots, x_M) = \frac{\det \left[\hat{f}_i^{(0)}; \hat{f}_i^{(1)}; \dots; \hat{f}_i^{(k)}; f_i(x_{p+1}); f_i(x_{p+2}); \dots; f_i(x_M) \right]}{\Delta(x_{p+1} \dots x_M)} \quad (97)$$

APPENDIX IV

MOMENT GENERATING FUNCTION FOR SEMICORRELATED AND NON-ZERO MEAN CHANNELS

In this appendix we use the joint probability distribution $P(\{\lambda_i\})$ derived in Sections IV-B and IV-C to calculate the corresponding moment generating function $g(z)$, (11). We start with the following Lemma.

Lemma 8: Let $F(\{\lambda_i\})$ be a function of λ_i , for $i = 1, \dots, N$, defined as

$$F(\{\lambda_i\}) = \mathcal{K} \prod_{i=1}^N f(\lambda_i) \Delta(\boldsymbol{\lambda}) \det \mathbf{P}(\boldsymbol{\lambda}) \quad (98)$$

where \mathcal{K} is a constant independent of λ_i and the $M \times M$ matrix \mathbf{P} , with $M \geq N$, has elements

$$P_{ij} = \begin{cases} p_i(\lambda_j) & j \leq N \\ q_{ij} & j > N \end{cases} \quad (99)$$

for some function $p_i(x)$. Then integrating F over λ we have

$$\prod_{j=1}^N \int_0^\infty d\lambda_j F(\{\lambda_i\}) = N! \det \tilde{\mathbf{P}} \quad (100)$$

where the $M \times M$ matrix $\tilde{\mathbf{P}}$ has elements

$$\tilde{P}_{ij} = \begin{cases} \int_0^\infty d\lambda f(\lambda) \lambda^{i-1} p_j(\lambda) & i \leq N \\ q_{ij} & i > N \end{cases} \quad (101)$$

Proof: We expand the two determinants as in (2) resulting in

$$\begin{aligned} \prod_{j=1}^N \int_0^\infty d\lambda_j F(\{\lambda_i\}) &= \sum_{\mathbf{a}} \sum_{\mathbf{b}} \text{sgn}(\mathbf{a}) \text{sgn}(\mathbf{b}) \prod_{i=1}^N \int_0^\infty d\lambda_i f(\lambda_i) \lambda_i^{a_i-1} p_{b_i}(\lambda_i) \prod_{i=N+1}^M q_{ib_i} \\ &= \sum_{\mathbf{a}} \sum_{\mathbf{b}} \text{sgn}(\mathbf{a}) \text{sgn}(\mathbf{b}) \prod_{i=1}^N \tilde{P}_{a_i b_i} \prod_{i=N+1}^M q_{ib_i} \\ &= N! \sum_{\mathbf{b}} \text{sgn}(\mathbf{b}) \prod_{i=1}^M \tilde{P}_{ib_i} \\ &= N! \det \tilde{\mathbf{P}} \end{aligned} \quad (102)$$

■

We now apply this lemma to the two cases of semicorrelated and non-zero mean channels analyzed in Sections IV-B and IV-C.

A. Moment generating function for semicorrelated channels

We can immediately apply the above lemma to calculate the moment generating function. We distinguish two cases:

1) $N = n_t \leq n_r = M$: In this case, the form of the matrix \mathbf{P} simplifies, since there are no q_{ij} terms (see (38)) and we thus have $f(\lambda) = (1 + \lambda)^z \lambda^{n_r - n_t}$ and $p_j(\lambda) = e^{-t_j \lambda}$, resulting in

$$g(z) = (-1)^{n_t(n_t-1)/2} \prod_{j=1}^{n_t} t_j^{n_r} \frac{\det \mathbf{L}}{\Delta(\mathbf{t})} \quad (103)$$

with the $n_t \times n_t$ matrix \mathbf{L} has elements

$$L_{ij} = \Psi(n_r - n_t + j, n_r - n_t + j + z + 1, t_i) \quad (104)$$

where $\Psi(a, b, z)$ is the confluent hypergeometric function [37], given by

$$\Psi(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty dt e^{-tz} t^{a-1} (1+t)^{b-a-1} \quad (105)$$

Note we absorbed the product of factorials in (38) into the Ψ function.

2) $M = n_t > n_r = N$: In this case we use (39) and set $f(\lambda) = (1 + \lambda)^z$, $q_{ij} = t_i^{j - n_r - 1}$ and $p_i(\lambda) = e^{-t_i \lambda}$, resulting in the following expression for $g(z)$

$$g(z) = (-1)^{n_t(n_t-1)/2} \prod_{i=1}^{n_t} t_i^{n_r} \frac{\det \mathbf{L}}{\Delta(\mathbf{t})} \quad (106)$$

with the $n_t \times n_t$ matrix \mathbf{L} has elements

$$L_{ij} = \begin{cases} t_i^{j-1} & 1 \leq j \leq n_t - n_r \\ \Psi(n_r - n_t + j, n_r - n_t + j + z + 1, t_i) & j > n_t - n_r \end{cases} \quad (107)$$

3) *Ergodic Capacity*: By employing methods similar to Section VI the expression for the ergodic mutual information of semicorrelated channels, which is analogous to (79) and in agreement with [17], [18]

$$E[I] = \sum_{j=1}^N \frac{\det \tilde{\mathbf{L}}_j}{\det \mathbf{L}} \quad (108)$$

where \mathbf{L} is given by (103) or (106) evaluated at $z = 0$ and $\tilde{\mathbf{L}}_j$ is given by the matrix \mathbf{L} , with the elements of the column $j + \max(0, n_t - n_r)$ substituted by

$$D_{ij} = \frac{(-1)^{M-n_t+j}}{(M - n_t + j - 1)!} \left[\frac{e^{t_i} E i(-t_i)}{t_i} \right]^{(M-n_t+j-1)} \quad (109)$$

with the superscript in the last term indicating the number of derivatives applied.

B. Moment generating function for channels with non-zero mean

Applying Lemma 8 to (48) we obtain the following expression for $g(z)$

$$g(z) = (-1)^{(N_0+N)(M-1)} \frac{\prod_{i=1}^{M-N} \left[\frac{(N+i-1)!}{(i-1)!} \right]}{\prod_{j=1}^{M-N_0} [(j-1)!]^2} \prod_{j=1}^{N_0} \left[\gamma_j^{N_0-M} e^{-\gamma_j} \right] \frac{\det \mathbf{L}}{\Delta(\boldsymbol{\gamma})} \quad (110)$$

\mathbf{L} is a $M \times M$ matrix with elements

$$L_{ij} = \begin{cases} \int_0^\infty d\lambda \lambda^{j-1} (1+\lambda)^z e^{-\lambda} I_0(2\sqrt{\gamma_i \lambda}) & i \leq N_0, j \leq N \\ (i+j-N_0-1)! \Psi(i+j-N_0-1, i+j-N_0+z, 1) & i > N_0, j \leq N \\ \gamma_i^{j-N-1} & i \leq N_0, j > N \\ [(i-N_0-1)!]^2 \delta_{i-N_0, j-N} & i > N_0, j > N \end{cases} \quad (111)$$

APPENDIX V

OTHER DIRECTIONS: APPLICATION OF UNITARY INTEGRALS TO UNITARY SPACE-TIME ENCODING

In this section we will briefly show another application of integration over unitary groups, where the character expansion and other results analyzed above may be applied, namely in the field of unitary space-time encoding. This is a scheme suggested recently by [24]–[27], in which the signal is encoded using unitary matrices across multiple transmit antennas and time-slots. The interest in this encoding method lies in its very good performance for large signal to noise ratios in the absence of any channel information at the receiver. [27]

In particular, we assume that the channel matrix \mathbf{G} is constant over T time-intervals and then changes completely. In this case the channel equation (5) needs to be modified to [24]

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \quad (112)$$

where \mathbf{G} is a $n_t \times n_r$ matrix with the channel coefficients from the transmitting to the receiving arrays, \mathbf{Z} is the $n_r \times T$ additive noise matrix, while \mathbf{Y} and \mathbf{X} are $n_r \times T$ and $n_t \times T$ dimensional matrices representing the output and input signals. The noise elements Z_{ij} are assumed to be Gaussian i.i.d. with unit variance, while \mathbf{G} is semicorrelated at the transmitter end, i.e. $E[G_{ia}G_{jb}] = \delta_{ij}T_{ab}$. Their instantaneous values are assumed to be unknown to both the transmitter and the receiver.

Recently [27] analyzed the mutual information of this problem, when the input distribution \mathbf{X} is unitary, i.e. when

$$\mathbf{X}\mathbf{X}^\dagger = \mathbf{I}_{n_t} \quad (113)$$

(but $\mathbf{X}^\dagger\mathbf{X} \neq \mathbf{I}_T$). One can thus express \mathbf{X} as

$$\mathbf{X} = \mathbf{J}_{n_t} \mathbf{U} \quad (114)$$

where \mathbf{U} is an $T \times T$ unitary matrix and \mathbf{J}_{n_t} is a $n_t \times T$ matrix with $J_{ab} = \delta_{ab}$. (114) includes only n_t out of T orthogonal T -dimensional vectors. To obtain the capacity of such encoding structures, it is essential to first calculate the received signal probability distribution, which involves an integration over all transmitted signals, which in this case are unitary matrices.

Since both \mathbf{G} and \mathbf{Z} are Gaussian matrices, the conditional distribution of \mathbf{Y} on \mathbf{X} , $p(\mathbf{Y}|\mathbf{X})$ will also be Gaussian, with density

$$\begin{aligned} p(\mathbf{Y}|\mathbf{X}) &= \frac{\exp\left(-\text{Tr}\left\{\mathbf{Y}\left[\mathbf{I}_T + \mathbf{X}^\dagger\mathbf{T}\mathbf{X}\right]^{-1}\mathbf{Y}^\dagger\right\}\right)}{\pi^{Tn_r}\det(\mathbf{I}_T + \mathbf{X}^\dagger\mathbf{T}\mathbf{X})^{n_r}} \\ &= \frac{\exp\left(-\text{Tr}\left\{\mathbf{Y}\left[\mathbf{I}_T - \mathbf{X}^\dagger\frac{\mathbf{T}}{\mathbf{I}_{n_t} + \mathbf{T}}\mathbf{X}\right]\mathbf{Y}^\dagger\right\}\right)}{\pi^{Tn_r}\det(\mathbf{I}_{n_t} + \mathbf{T})^{n_r}} \end{aligned} \quad (115)$$

where the second equation results from \mathbf{X} being unitary, (113). The received signal distribution is

$$p(\mathbf{Y}) = \int d\mathbf{X} p(\mathbf{Y}|\mathbf{X}) \quad (116)$$

[27] obtained a closed form expression for the case $T_{ab} = \delta_{ab}$ using different methods. Here we will obtain a closed form expression for a general positive definite \mathbf{T} .

We can rewrite the above integral over \mathbf{X} using (114) as

$$p(\mathbf{Y}) = \int D\mathbf{U} \frac{\exp\left(-\text{Tr}\left\{\mathbf{Y}\left[\mathbf{I}_T - \mathbf{U}^\dagger\tilde{\mathbf{T}}\mathbf{U}\right]\mathbf{Y}^\dagger\right\}\right)}{\pi^{Tn_r}\det(\mathbf{I}_{n_t} + \mathbf{T})^{n_r}} \quad (117)$$

where

$$\tilde{\mathbf{T}} = \mathbf{J}_{n_t} \frac{\mathbf{T}}{\mathbf{I}_{n_t} + \mathbf{T}} \mathbf{J}_{n_t}^\dagger \quad (118)$$

We immediately see that the integral in (117) is of the form of (29), which allows us to immediately obtain the result. We are interested in the case $T \geq n_t$. [27] Defining the T -dimensional vector $\tilde{\mathbf{t}}$ so that $\tilde{t}_i = t_i = T_i/(1 + T_i)$ for $i = 1, \dots, n_t$, where T_i are the eigenvalues of \mathbf{T} , and using (37) we have

$$p(\mathbf{Y}) = \frac{\prod_{p=1}^{T-1} p!}{\prod_{j=1}^{n_t} (1 + T_j)} \frac{\det\left(e^{y_i \tilde{t}_j}\right)}{\Delta(\mathbf{y})\Delta(\tilde{\mathbf{t}})} \exp\left(-\sum_i y_i\right) \quad (119)$$

where \mathbf{y} is the T -dimensional vector of eigenvalues of $\mathbf{Y}^\dagger\mathbf{Y}$. We now need to take $\tilde{t}_i \rightarrow 0$ for $i = n_t + 1, \dots, T$. Also, if $T > n_r$ we need to set $y_i = 0$ for $i = n_r + 1, \dots, T$. These limits can be taken by using Lemma 6. The final result is

$$p(\mathbf{Y}) = \frac{\prod_{p=T-n_t}^{T-1} p!}{\prod_{j=1}^{n_t} (1 + T_j) \prod_{i=1}^{T-Q-1} i!} \frac{\det \mathbf{L}}{\Delta(\mathbf{y}) \prod_{i=1}^Q y_i^{T-Q} \Delta(\mathbf{t}) \prod_{j=1}^{n_t} t_i^{T-n_t}} \exp\left(-\sum_{i=1}^Q y_i\right) \quad (120)$$

where $Q = \min(T, n_r)$ and the T -dimensional matrix \mathbf{L} has elements

$$L_{ij} = \begin{cases} e^{y_i t_j} & i \leq Q, j \leq n_t \\ y_i^{j-n_t-1} & i \leq Q, j > n_t \\ t_j^{i-Q-1} & i > Q, j \leq n_t \\ (i - n_r - 1)! \delta_{i-n_r, j-n_t} & i > Q, j > n_t \end{cases} \quad (121)$$

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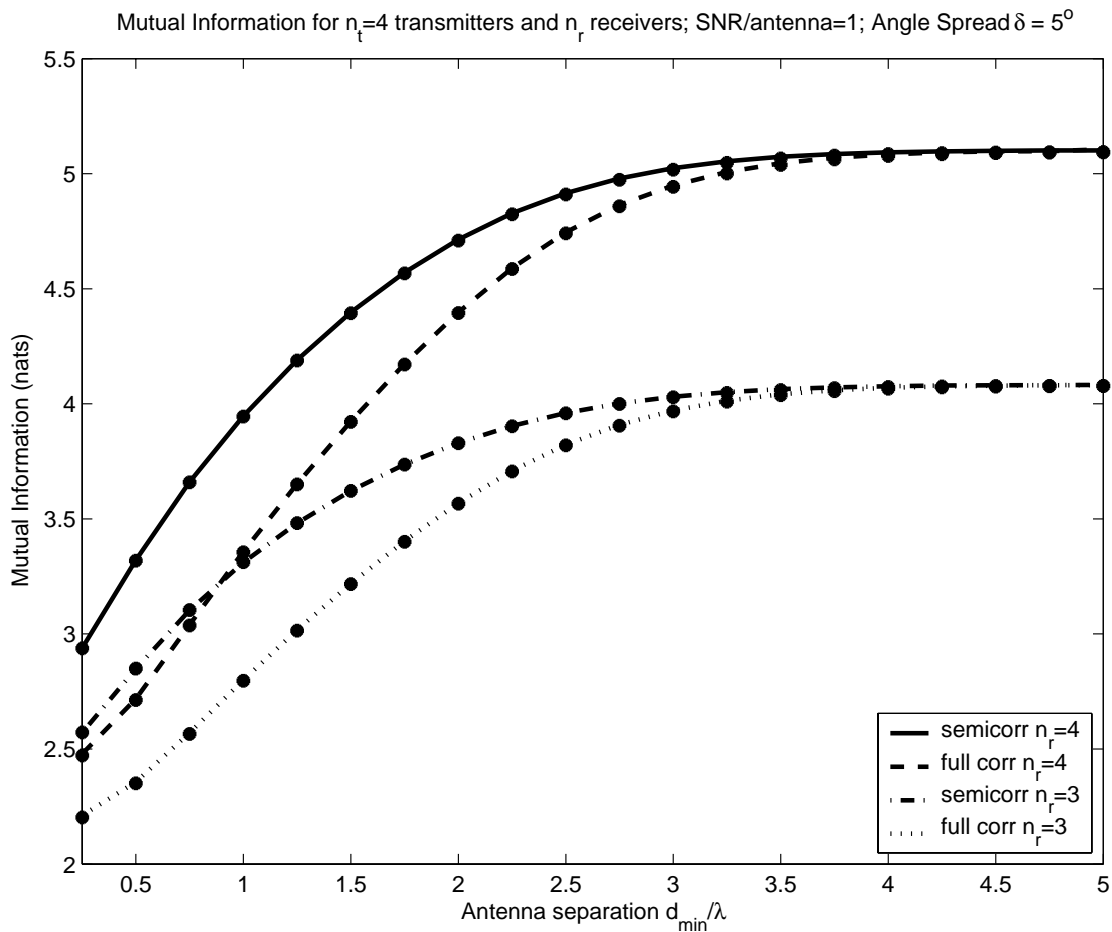


Fig. 1. Plot of mutual information for a $n_t = 4$ -transmitting-antenna and a n_r -receiving-antenna array pair for $n_r = 3, 4$. Both transmitter and receiver arrays are uniformly spaced with d_{min} spacing between neighboring antennas. The solid and dashed curves have correlations between elements on both antenna arrays given by (82) with angle spread $\delta = 10^\circ$. Once the eigenvalues of the correlation matrix are evaluated, they are inserted in (79), which gives the ergodic mutual information. The receiving array of the other two curves (dotted and dot-dashed) has no antenna correlations. In this case (108) in Appendix IV-A is used to calculate the mutual information. We see that at large antenna separation the antenna correlations become negligible and the correlated cases approach the semicorrelated ones. The asterisks represent the values of the simulated average capacities and have been added as a cross-check.

Outage mutual information for $n_t = 4$ transmitters and n_r receivers for 2 separations d_{min} ; SNR/antenna=1; Angle Spread $\delta = 5^\circ$

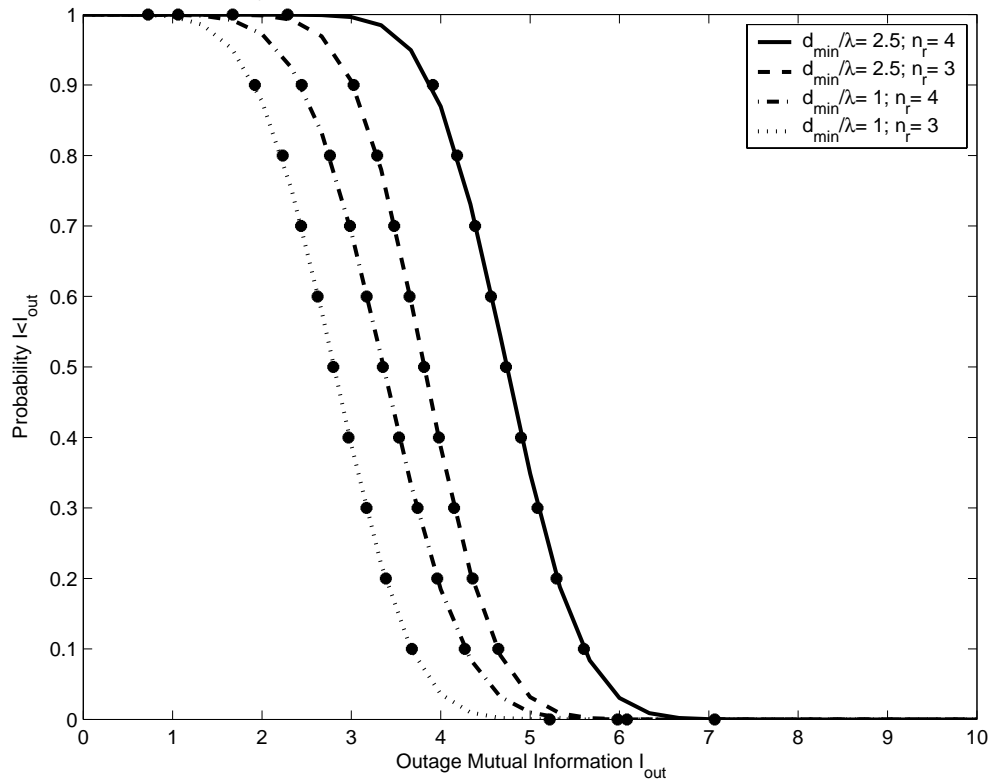


Fig. 2. Plot of the outage mutual information for a $n_t = 4$ -transmitting-antenna and a n_r -receiving-antenna array pair for $n_r = 3, 4$ and two different array spacings d_{min} . The other parameters used are the same as in Fig. 1. The curves correspond to the values of probability versus outage capacity, which were obtained by numerically integrating (10). The asterisks result from Monte-Carlo simulations and have been added as verification of the numerical integral.