

A Note on Double Antenna Diagonal Space-Time Codes

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Abstract

Optimizing the diversity product of two-antenna diagonal space time codes can be phrased as finding, for a given integer L , the maximum over all positive integers $u < L$ of the minimum over all positive integers $x < L$ of the expression $|\sin(\pi x/L) \sin(\pi x u/L)|$. We establish a relationship between this optimization problem and the elementary theory of continued fractions. In particular, we show that the u that maximizes the diversity product must have the property that u/L cannot be “approximated too well” by fractions a/b with $b < L$. Inspired by the well-known fact that quotients of Fibonacci numbers have such a property, we study the case where L is a Fibonacci number and derive bounds for the best diversity product.

1 Introduction

In [1] the authors introduced a differential unitary space time coding scheme which is well-suited for multiple antenna wireless transmission in which the fading channel is neither known to the sender nor to the receiver. The transmission proceeds as follows: Let \mathcal{V} be a set of L unitary $M \times M$ -matrices, where M is the number of receiving antennas. We assume that the fading coefficients are constant over $T = 2M$ consecutive uses of the channel. This way, the matrices transmitted over the network are $2M \times M$ -matrices, and they consist of two blocks of $M \times M$ -matrices which are obtained in the following way. Suppose that the matrices in \mathcal{V} are V_1, \dots, V_L , and that the stream of integers τ_1, τ_2, \dots is to be transmitted over the system, where $\tau_i \in \{1, \dots, L\}$. In the first transmission, the M transmit antennas send the $2M \times M$ -matrix $\begin{pmatrix} I \\ V_{\tau_1} \end{pmatrix}$ over the antennas, where I is the $M \times M$ -identity matrix. Suppose that at round i the antennas transmit the matrix $\begin{pmatrix} W_{i-1} \\ W_i \end{pmatrix}$. Then the matrix

transmitted at round $i + 1$ is $\binom{W_i}{W_i V_{\tau_{i+1}}}$. The set \mathcal{V} is also called a differential space time code [1].

The study of the pairwise probability of error of the maximum likelihood decoder for this system reveals that this probability is lower the larger the *diversity product*

$$\zeta(\mathcal{V}) := \min_{V, W \in \mathcal{V}, V \neq W} |\det(V - W)|$$

is [1]. The differential space time code design problem is thus to make L , the size of \mathcal{V} , as large as possible while maintaining the diversity product. Hochwald and Sweldens introduced in [1] a particular class of differential space time codes which consist of diagonal matrices. More precisely, let η be a primitive L th root of unity, and let u_1, \dots, u_M be integers. Then the ℓ th element of \mathcal{V} , $\ell = 0, \dots, L - 1$, is the diagonal matrix

$$\begin{pmatrix} \eta^{u_1 \ell} & 0 & 0 & \cdots & 0 \\ 0 & \eta^{u_2 \ell} & 0 & \cdots & 0 \\ 0 & 0 & \eta^{u_3 \ell} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \eta^{u_M \ell} \end{pmatrix}.$$

The diversity product of this set (which in fact forms a cyclic group) is

$$\zeta(\mathcal{V}) = \min_{1 \leq \ell < L} |1 - \eta^{u_1 \ell}| \cdots |1 - \eta^{u_M \ell}|.$$

To obtain the largest possible diversity product, the u_i have to be chosen appropriately. An efficient procedure for this task is, however, unknown.

In this paper we will discuss the problem of optimizing the u_i for the case of two transmit antennas, i.e., $M = 2$. Although there are other very good codes known in this setting [2, 3], the diagonal codes with two transmitter antennas are still of some practical interest because they have very simple and efficient encoding and decoding [4].

It is easy to see that one can choose u_1 to be 1, without loss of generality. The diversity product then only depends on L and u_2 . To save notation, we will replace u_2 by u . We define

$\zeta(u, L) := \min_{1 \leq x < L} |1 - e^{2\pi i x/L}| |1 - e^{2\pi i u x/L}|$. Then we have

$$\zeta(u, L) = 4 \min_{1 \leq x < L} \left| \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x u}{L}\right) \right|.$$

Further, we define $\zeta(L) := \max_{1 \leq u < L} \zeta(u, L)$.

The problem we are interested in is that of an efficient computation of an integer u such that $\zeta(u, L) = \zeta(L)$. (By efficient we mean an algorithm that runs in time polynomial in $\log(L)$, say.) This problem seems to be hard. What we can show is a relaxation of the problem: for the sequence F_n of Fibonacci numbers we show that $\zeta(F_{n-2}, F_n) \geq c\zeta(F_n)$ for an absolute constant c (which is roughly 0.38). In fact, based on extensive computations and some extremality properties of Fibonacci numbers, we *conjecture* that $\zeta(F_n) = \zeta(F_{n-2}, F_n)$.

The Fibonacci numbers are interesting from a practical point of view since the quotient of two consecutive such numbers tends to $(1 + \sqrt{5})/2$ which is slightly larger than 2. If the application requires working with a block size L which is a power of 2, then one can virtually work with a larger block size $L' = F_n$, where F_n is the smallest Fibonacci number larger than L . This would also provide a limited way of error detection, since any decoded matrix whose index is larger than L can be declared to be in error.

To achieve our goals we will first relate the problem of approximating $\zeta(u, L)$ to the continued fraction expansion of u/L . The starting point for this is the lattice description of an approximation of $\zeta(u, L)$, as was done in [4]. We will prove in the next section that $\zeta(u, L)$ can be well-approximated by a function on a certain lattice depending on u and L . Then we will show that this approximation can be efficiently computed using the continued fraction expansion of u/L , see Theorem 1. Let us recall the basic definitions: Applying the Euclidean algorithm to the pair (u, L) , we obtain a sequence of *partial quotients* $q_0, q_1, q_2, \dots, q_t$, $q_t \geq 2$, where $u = q_0 L + r_0$, $L = q_1 r_0 + r_1, \dots, r_{t-2} = q_t r_{t-1}$. (Note that $q_0 = 0$ for $u < L$.) Conversely, the sequence q_0, q_1, \dots, q_t uniquely determines u/L in the following way: Define $Q_{-2} = 1, Q_{-1} = 0$ and $P_{-2} = 0, P_{-1} = 1$, and $Q_\ell := q_\ell Q_{\ell-1} + Q_{\ell-2}$, $P_\ell := q_\ell P_{\ell-1} + P_{\ell-2}$ for $\ell \geq 0$. Then $P_t/Q_t = u/L$. The fractions P_ℓ/Q_ℓ are called *convergents* of u/L . They have various interesting properties of which we shall use quite a few. Because q_0, \dots, q_t uniquely describes u/L , it is customary to define $[q_0, q_1, \dots, q_t]$ as u/L .

It turns out that, in an approximate sense, $\zeta(u, L)$ is the larger the smaller the maximum of the partial quotients is (see Proposition 2). Therefore, to optimize $\zeta(u, L)$ we need to control the partial

quotients of u/L . This does not seem to be an easy task for general L . However, if L is a Fibonacci number F_n , the partial quotients of F_{n-2}/F_n are all one (except for the last one which is 2). This connection is exploited in Section 3.

2 Approximating the sin-function on lattices

We start this section by proving a general theorem on minimization of trigonometric functions on a two-dimensional lattice.

Lemma 1. *Let L be an integer, and Λ be a lattice in \mathbb{R}^2 which contains $(0, L)$. Then we have the following:*

$$\alpha^2 \frac{\pi^2}{L^2} \mu(\Lambda) \leq \min_{(x,y) \in \Lambda, 0 < |x|, |y| < L} \left| \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \right| \leq \frac{\pi^2}{L^2} \mu(\Lambda),$$

where $\mu(\Lambda) := \min_{(x,y) \in \Lambda, 0 < |x|, |y| < L} |xy|$, and $\alpha = 1 - \pi^2/24 \sim 0.5888$.

Proof. By symmetry, we may assume that $0 < x \leq L/2$. Then we have

$$\frac{\pi x}{L} \left(1 - \frac{\pi^2}{24}\right) \leq \frac{\pi x}{L} \left(1 - \frac{\pi^2 x^2}{6L^2}\right) \leq \left| \sin\left(\frac{\pi x}{L}\right) \right| \leq \frac{\pi x}{L}.$$

(The first inequality follows from $x \leq L/2$, the second from $\sin(x) \geq x - x^3/6$ for $x \geq 0$, and the third from $\sin(x) \leq x$ for $x \geq 0$.) By symmetry, we may assume that $y \geq 0$. There are two possibilities for y : either $y \leq L/2$, or $L/2 < y < L$. In the first case, we deduce as above that

$$\frac{\pi y}{L} \left(1 - \frac{\pi^2}{24}\right) \leq \left| \sin\left(\frac{\pi y}{L}\right) \right| \leq \frac{\pi y}{L}.$$

In the second case, we observe that $\sin(\pi y/L) = \sin(\pi(L-y)/L)$, and that $L-y < L/2$. Therefore,

$$\frac{\pi(L-y)}{L} \left(1 - \frac{\pi^2}{24}\right) \leq \left| \sin\left(\frac{\pi y}{L}\right) \right| \leq \frac{\pi(L-y)}{L}.$$

This shows that

$$\alpha^2 \frac{\pi^2}{L^2} \eta \leq \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \leq \frac{\pi^2}{L^2} \eta,$$

where

$$\eta = \min_{(x,y) \in \Lambda, 0 < |x|, |y| < L} \min(|xy|, |x(L-y)|).$$

Note that since $(0, L)$ is in the lattice, the point $(-x, L - y)$ also belongs to the lattice if (x, y) does. Hence, $\eta = \mu(\Lambda)$ and we are done. \square

By definition, $\zeta(u, L)$ is the minimum of the function $|\sin(\pi x/L) \sin(\pi y/L)|$, where $y = xu \pmod L$. In view of the previous theorem, it makes sense to consider the lattice $\Lambda = \Lambda(u, L) = \mathbb{Z}(1, u) + \mathbb{Z}(0, L)$. Hence, Λ is the set of vectors of the form $(x, xu + \mathbb{Z}L)$ where x runs over the integers. In this case, the quantity $\mu(\Lambda)$ defined in the previous theorem has an interesting expression in terms of the continued fraction expansion of u/L . In the following, we define

$$\mu(u, L) := \mu(\Lambda(u, L)) = \min_{(x,y) \in \Lambda(u,L), 0 < |x|, |y| < L} |xy|.$$

Proposition 1. *Let $P_1/Q_1, \dots, P_t/Q_t = u/L$ be the successive convergents of u/L . Then we have*

$$\mu(u, L) = \min_{\ell=1}^{t-1} Q_\ell |Q_\ell u - P_\ell L|.$$

Proof. We first show that

$$\min_{(x,y) \in \Lambda(u,L), 0 < |x|, |y|} |xy| = \min_{\ell=1}^{t-1} Q_\ell |Q_\ell u - P_\ell L|. \quad (1)$$

This will indeed imply the assertion, if we show that Q_ℓ and $|Q_\ell u - P_\ell L|$ are both smaller than L . It is clear that Q_ℓ is less than L since the sequence of the Q_i is monotonically increasing and $Q_t = L$. Further, since $|u/L - P_\ell/Q_\ell| < 1/Q_\ell^2$ by [5, p. 43,(9)] that $|Q_\ell u - P_\ell L| < L/Q_\ell \leq L$.

Suppose now that $Q_\ell \leq x < Q_{\ell+1}$. Then, by the approximation theorem [5, p. 52, Satz 16] we have $|u/L + A/x| \geq |u/L - P_\ell/Q_\ell|$ for *any* integral A . Hence, $x|ux + \mathbb{Z}L| \geq Q_\ell |Q_\ell u - P_\ell L|$ which means that for the minimization, we only need to consider the Q_ℓ 's. This proves the proposition. \square

Combination of the last two results yields

Theorem 1. *Let u and L be positive integers $u < L$. Suppose that $P_1/Q_1, \dots, P_t/Q_t = u/L$ are the convergents of u/L , and let $\mu(u, L) := \min_{\ell=1}^{t-1} Q_\ell |Q_\ell u - P_\ell L|$. Then we have*

$$\alpha^2 \frac{\pi^2}{L^2} \mu(u, L) \leq \zeta(u, L) \leq \frac{\pi^2}{L^2} \mu(u, L),$$

where α is defined in Lemma 1.

In order to obtain the best estimates for $\zeta(L)$, we need to know when $\mu(u, L)$ is maximal. This seems to be a hard problem in general. However, some partial assertions can be made.

Proposition 2. *Let $u/L = [0, q_1, \dots, q_t]$ and $w/L = [0, p_1, \dots, p_s]$. Suppose that $\max_i q_i > \max_j p_j + 1$. Then $\mu(u, L) < \mu(w, L)$.*

Proof. Let $P_1/Q_1, \dots, P_t/Q_t$ be the convergents of u/L . We will use the following well-known result [5, p. 43,(8)]

$$\left| \frac{u}{L} - \frac{P_\ell}{Q_\ell} \right| = \frac{1}{Q_\ell(\xi_{\ell+1}Q_\ell + Q_{\ell-1})},$$

where $\xi_{\ell+1} = [q_{\ell+1}, q_{\ell+2}, \dots, q_t]$ and $[0, q_1, \dots, q_t] = u/L$. Let ℓ be such that $q_{\ell+1}$ is the maximum of the q_i . Then we have

$$Q_\ell |Q_\ell u - P_\ell L| = L Q_\ell^2 \left| \frac{u}{L} - \frac{P_\ell}{Q_\ell} \right| = \frac{L}{\xi_{\ell+1} + Q_{\ell-1}/Q_\ell}.$$

Similarly, if $[0, p_1, \dots, p_r] = w/L$, $A_1/B_1, \dots, A_r/B_r = w/L$ are the convergents of w/L , and s is such that p_{s+1} is the maximum of the p_i , then we have

$$B_s |B_s w - A_s L| = \frac{L}{\eta_{s+1} + B_{s-1}/B_s},$$

where $\eta = [p_{s+1}, \dots, p_r]$. It thus remains to show that

$$\xi_{\ell+1} + Q_{\ell-1}/Q_\ell > \eta_{s+1} + B_{s-1}/B_s.$$

This is easy to see: The assumption that $q_{s+1} \geq p_{s+1} + 2$ implies that $\xi_{\ell+1} \geq q_{s+1} \geq p_{s+1} + 2$, and this, together with $\eta_{s+1} \leq p_{s+1} + 1/2$ and $B_{s-1}/B_s < 1$ yields

$$\xi_{\ell+1} + \frac{Q_{\ell-1}}{Q_\ell} > p_{s+1} + 2 > p_{s+1} + \frac{1}{2} + 1 > \eta_{s+1} + \frac{B_{s-1}}{B_s}.$$

This completes the proof. □

In view of the previous result, we should look for u 's for which the maximum partial quotient of u/L is as small as possible. In general, the best strategy for this seems to be to try out all the possible u 's. However, for specific L 's this problem can be solved efficiently. This is demonstrated in the next section.

3 The Fibonacci Case

Let F_n be the n th Fibonacci number defined by the recursion $F_{k+1} = F_k + F_{k-1}$, and the initial conditions $F_0 = F_1 = 1$. The aim of this section is to show the following.

Theorem 2. *We have*

$$\max_{1 \leq u < F_n} \mu(u, F_n) = \mu(F_{n-2}, F_n) = F_{n-2}.$$

In particular,

$$\alpha^2 \frac{\pi^2}{F_n^2} F_{n-2} \leq \zeta(F_n) \leq \frac{\pi^2}{F_n^2} F_{n-2},$$

where $\alpha = 1 - \pi^2/24$.

For the proof we need some preparations.

Lemma 2. *Let ℓ, n be nonnegative integers with $\ell \leq n$, and $n \geq 2$. Then we have*

$$(1) \quad F_\ell F_n - F_{\ell+2} F_{n-2} = (-1)^\ell F_{n-\ell-3}.$$

$$(2) \quad F_{n-1} F_\ell - F_{\ell-1} F_n = (-1)^\ell F_{n-\ell-1}.$$

Proof. Both assertions can be proved via induction. Since the proofs are similar, we will only focus on the first part. Let n be fixed. The assertion is easily shown to be true for $\ell = 0$. We now use induction on ℓ . Noting that

$$\begin{aligned} F_{\ell+1} F_n - F_{\ell+3} F_{n-2} &= F_\ell F_n + F_{\ell-1} F_n - F_{\ell+2} F_{n-2} - F_{\ell+1} F_{n-2} \\ &= (-1)^\ell F_{n-\ell-3} + (-1)^{\ell-1} F_{n-\ell-2} \\ &= (-1)^{\ell+1} F_{n-\ell-4}. \end{aligned}$$

This proves the assertion. □

Corollary 1. *Let $n \geq 2$ be an integer. Then we have*

$$\min_{1 \leq \ell < n} F_\ell |F_{n-1} F_\ell - F_{\ell-1} F_n| = F_{n-2}.$$

Proof. By Lemma 2(2), the above quantity equals $\min_\ell F_\ell F_{n-\ell-1}$. For $\ell = 1$ this quantity is F_{n-2} , so we only need to show that $F_\ell F_{n-\ell-1} - F_{n-2} \geq 0$ for $1 \leq \ell < n$. Let us denote by a_ℓ the expression

$F_\ell F_{n-\ell-1} - F_{n-2}$. We will show that for all $1 \leq \ell \leq (n-1)/4$ we have

$$a_{2\ell+2} < a_{2\ell}, \quad \text{and} \quad a_{2\ell-1} < a_{2\ell+1}.$$

To this end, note that

$$a_{2\ell} - a_{2\ell+2} = F_{2\ell} F_{n-2\ell-1} - F_{2\ell+2} F_{n-2\ell-3}.$$

Using Lemma 2(1), we see that the right hand side equals $(-1)^{2\ell} F_{n-4\ell-4} > 0$ if $n \geq 5$. (We use the lemma with n replaced by $n-2\ell-1$ and ℓ replaced by 2ℓ . It is easily checked that the assumptions of the lemma are valid since $n \geq 5$.) The assertion on $a_{2\ell-1}$ can be proved similarly.

The monotonicity properties of the a_ℓ together with the easily verified fact $a_{n-1-\ell} = a_\ell$ show that $a_\ell \geq \min(a_1, a_p)$, where p is the largest even integer $\leq n/2$. So, p has one of the values $(n-1)/2, (n-1)/2-1, (n-2)/2$, or $(n-4)/2$. In all these cases, one can check via induction that $a_p \geq 0$. Since $a_1 = 0$, we thus see that $a_\ell \geq 0$ for $1 \leq \ell \leq n-2$, provided that $n \geq 5$. For $n < 5$ the assertion of the corollary can be proved by hand. \square

We are now ready to prove Theorem 2.

Proof of Theorem 2. We first show that $\mu(u, F_n) \leq F_{n-2}$ for any u with $1 \leq u < F_n$. Since $\mu(u, F_n) = \mu(F_n - u, F_n)$, we may assume that $u \geq F_n/2$.

Suppose first that u is such that u/F_n has a partial quotient $q_{\ell+1} \geq 3$, and let P_ℓ/Q_ℓ be the ℓ th convergent of u/F_n . Then, by [5, p. 43,(8)], we have

$$\left| \frac{u}{F_n} - \frac{P_\ell}{Q_\ell} \right| \leq \frac{1}{Q_\ell(q_{\ell+1}Q_\ell + Q_{\ell-1})} < \frac{1}{3Q_\ell^2}.$$

Hence, $Q_\ell |uQ_\ell - P_\ell F_n| < F_n/3 \leq F_{n-2}$, where the last inequality is easily proved by induction for all $n \geq 3$. (If we assume $n \geq 4$, then the inequality is even strict.)

Suppose now that u/F_n has the partial quotients $0 = q_0, q_1, \dots, q_t$, and that q_1, \dots, q_t are either 1 or 2. Assume that $q_1 = \dots = q_\ell = 1, q_{\ell+1} = 2$ for some $\ell \geq 1$. (This possible since we have assumed that $u \geq F_n/2$.) If $q_{\ell+1}$ is the last partial quotient, then it is easily seen that $u = F_{n-1}$. We will deal with this case later. So, suppose that $q_{\ell+1}$ is not the last partial quotient.

Consider the rational number $\xi_{\ell+1}$ with the partial quotients $[q_{\ell+1}, q_{\ell+2}, \dots, q_t]$. Since by assumption all the q_i are in the set $\{1, 2\}$, we have that $\xi_{\ell+1} \geq [2, 2, \dots, 2] > [2, 2, 2, \dots] = \sqrt{2} + 1$.

By [5, p. 43,(8)] we have

$$\left| \frac{u}{F_n} - \frac{P_\ell}{Q_\ell} \right| = \frac{1}{Q_\ell(\xi_{\ell+1}Q_\ell + Q_{\ell-1})}.$$

By our assumption on the partial quotients of u/F_n , we have $Q_i = F_i$ for $i \leq \ell$. So, we have

$$\begin{aligned} Q_\ell |uQ_\ell - F_n P_\ell| &= F_n \frac{F_\ell}{\xi_{\ell+1}F_\ell + F_{\ell-1}} \\ &\leq F_n \frac{F_\ell}{2F_\ell + F_{\ell-1} + (\sqrt{2} - 1)F_\ell} \\ &= \frac{F_n F_\ell}{F_{\ell+2} + (\sqrt{2} - 1)F_\ell}. \end{aligned}$$

To prove that the latter is at most F_{n-2} , it suffices to show that

$$F_\ell F_n - F_{n-2} F_{\ell+2} < (\sqrt{2} - 1) F_\ell F_{n-2}.$$

By Lemma 2(1) the left hand side equals $(-1)^\ell F_{n-\ell-3}$ which shows that the inequality is trivially valid for odd ℓ . For even ℓ , it suffices to show the assertion for $\ell = 2$, since $F_{n-\ell-3}$ decreases with ℓ , and the right hand side increases with ℓ . Thus, we are left with the assertion $F_{n-5} < 2(\sqrt{2} - 1)F_{n-2}$. This is easily proved by induction.

The only case for u left to deal with is that in which u/F_n has the partial quotients $[0, 1, 1, \dots, 1, 2]$. In this case $u = F_{n-1}$, and the convergents of u/F_n are the fractions $F_{\ell-1}/F_\ell$, $\ell = 1, \dots, n$. So, to finish the proof of the theorem, we need to show that

$$\min_{1 \leq \ell < n} F_\ell |F_{n-1} F_\ell - F_{\ell-1} F_n| = F_{n-2}.$$

But this is precisely what was proved in Corollary 1, and we are done. \square

Although we have determined upper and lower bounds for $\zeta(F_n)$, the question about its exact value remains open. Based on extensive calculations and motivated by the extremal properties of the Fibonacci numbers, we conjecture that in fact

$$\zeta(F_n) = \zeta(F_{n-2}, F_n) = \left| \sin\left(\frac{\pi}{F_n}\right) \sin\left(\frac{\pi F_{n-2}}{F_n}\right) \right|.$$

4 Conclusions and further directions

In light of Proposition 2 and Theorem 1, we know that for a give L the diversity product is expected to be maximized for a u for which the partial quotients of u/L are small. In general, we do not know how to generate a u with smallest partial fractions in time that is polynomial in, say, $\log(L)$. For specific sequences of L 's however, this is quite possible as was shown in this paper for Fibonacci numbers. One can now reverse the problem and start with the partial quotients themselves, keeping their maximum small. An example is $[0, 1, \dots, 1, 2, 1, \dots, 1, 2]$. It is easily proved by induction that this is the continued fraction expansion of the quotient $(F_{t+1} + F_{t+1-i}F_{i-1})/(F_t + F_{t-i+1}F_{i-2})$, where i is the index of the first partial quotient equal to 2, and t is the index of the last partial quotient.

It is tempting to think that the continued fraction expansion of u/L is the ultimate key to the optimization problem for the diversity product. In fact, one may think that the u that maximizes $\mu(u, L)$ also maximizes $\zeta(u, L)$. Even though we have conjectured that this is true if L is a Fibonacci number, this is not true for general L . The smallest counterexample is $L = 76$, which was provided to us by Peter Oswald [6]. We have $\mu(76) = \mu(47, 76)$ and the continued fraction expansion of $47/76$ is $[0, 1, 1, 1, 1, 1, 1, 3]$. However, $\zeta(76) = \zeta(23, 76) \sim 0.3317$ and the continued fraction expansion of $23/76$ is $[0, 3, 3, 3, 2]$. However, the $\zeta(47, 76) \sim 0.3153$ which is very close to the optimal value. This may suggest that the constant α in Lemma 1 is far from optimal. In fact, based on lengthy computations, we think that this is indeed true, and that if u maximizes $\mu(u, L)$ then $\zeta(u, L)$ is very close to its optimal value.

Finally we mention that the methods of this paper may be applicable to solve another space time code design problem. In [7] and [8] the authors construct unitary space time codes from 2-dimensional representations of generalized Quaternion groups. To obtain such codes for more than two antennas, one has to use direct sums of irreducible representations. The problem that arises is which automorphisms of the group to use. This is an exact analogue of our problem since we the integer u can be identified with an automorphism of the cyclic group with L elements, if u and L are co-prime (if not, then the diversity product is zero).

References

- [1] B. Hochwald and W. Sweldens, "Differential unitary space time modulation," tech. rep., Bell Laboratories, Lucent Technologies, Mar. 1999. Also submitted to *IEEE Trans. Comm.* Downloadable

from <http://mars.bell-labs.com>.

- [2] S. M. Alamouti, “A simple transmitter diversity scheme for wireless communications,” *IEEE J. Sel. Area Comm.*, pp. 1451–1458, 1998.
- [3] V. Tarokh and H. Jafarkhani, “A differential detection scheme for transmit diversity,” *to appear in J. Sel. Area Comm.*, 2000.
- [4] K. L. Clarkson, W. Sweldens, and A. Zheng, “Fast multiple antenna differential decoding,” tech. rep., Bell Laboratories, Lucent Technologies, October 1999. Downloadable from <http://mars.bell-labs.com>.
- [5] O. Perron, *Die Lehre von den Kettenbrüchen*. Teubner Verlag, Leipzig und Berlin, 1913.
- [6] P. Oswald, “Private communication.” 2000.
- [7] B. Hughes, “Differential space-time modulation,” *submitted to IEEE Trans. Info. Theory*, 1999.
- [8] B. Hassibi, B. Hochwald, A. Shokrollahi, and W. Sweldens, “Representation theory for high-rate multiple-antenna code design,” tech. rep., Bell Laboratories, Lucent Technologies, 2000. Downloadable from <http://mars.bell-labs.com>.